Minimum Cost Flow by Successive Shortest Paths

- **Initialize to the 0 flow**
- **Repeat**
  - **Send flow along a shortest path in G_f**

Comments:

- **Correctly computes a minimum-cost flow**
- **Not polynomial time.**
- **Simple bound of $O(nmCU)$ time.**

Pseudoflow

Pseudoflow: A pseudoflow is a function on the edges of a graph satisfying

$$0 \leq f(v,w) \leq u(v,w) \;\; \forall (v,w) \in E$$

- Given a pseduflow f, we define the "excess" at v as

$$e(v) = b(v) + \sum_{w \in V} f(w,v) - \sum_{w \in V} f(v,w).$$

- If $e(v) = 0 \;\; \forall v \in V$, then a pseudoflow is a flow.
- We define reduced cost optimality of a pseudoflow f as

$$\exists \pi \mathbf{s.t.} c^{\pi}(v,w) \geq 0 \forall (v,w) \in G_f$$

Strategy: Maintain an f and π such that f is a pseudoflow satisfying reduced cost optimality. Work to make f a flow. When f is a flow, you know it is optimal.

How do you initialize?

- **You can assume that** $c(v, w) \geq 0 \forall (v, w) \in E$ **. Then the 0-flow satisfies reduced cost optimality.**
- **But what if the assumption doesn't hold?**

How do you initialize?

- **You can assume that $c(v, w) \geq 0 \forall (v, w) \in E$. Then the 0-flow satisfies reduced cost optimality.**
- **But what if the assumption doesn't hold?**

- **Set $f(v, w) = u(v, w)$ for all edges with $c(v, w) < 0$.**
- **Now, all edges in G_f , satisfy $c^{\pi}(v, w) \geq 0$.**
- **Update $e(v)$ accordingly.**

Successive Shortest Paths for Minimum Cost Flow

Successive Shortest Path

$f = 0; \quad \Pi = 0$
$e(v) = b(v)\ \forall v \in V$
Initialize $E = \{v : e(v) > 0\}$ **and** $D = \{v : e(v) < 0\}$
while $E \neq 0$
  Pick a node $k \in E$ **and** $\ell \in D$
  Compute $d(v)$**, shortest path distances from** k **in** G_f
    w.r.t. edge distances c^π**.**
  Let P **be a shortest path from** k **to** ℓ**.**
  Set $\pi = \pi - d$
  Let $\delta = \min\{e(k), -e(\ell), \min\{u_f(v,w) : (v,w) \in P\}\}$
  Send δ **units of flow on the path** P
  Update f**,** G_f**,** E**,** D **and** c^π**.**

Correctness of successive shortest path algorithm

Lemma: Let f be a pseudoflow satisfying reduced cost optimality with respect to π. Let $d(v)$ be the shortest path distance from some node s to v in G_f with respect to c^π. Then

- f satisfies reduced cost optimality with respect to $\pi' = \pi - d$.
- $c^{\pi'}(v, w) = 0$ if (v, w) is on a shortest path from s to some other node.

Correctness of successive shortest path algorithm

Corollary: **After each iteration of the successive shortest paths algorithm, f satisfies reduced cost optimality.**

But still not necessarily polynomial.

Use Capacity Scaling on top of shortest path algorithm

Def:

$$G_f(\Delta) = \{(v,w) \in G_f : u_f(v,w) \geq \Delta\}$$

Capacity Scaling Algorithm for Minimum Cost Flow

Successive Shortest Path

- $f = 0; \quad \pi = 0$
- $e(v) = b(v)\ \forall v \in V$
- $\Delta = 2^{\lfloor U \rfloor}$
- **while** $\Delta \geq 1$
  - **(**Δ **scaling phase)**
  - **for every edge** $(v, w) \in G_f$
    - **if** $u_f(v, w) \geq \Delta$ **and** $c^\pi(v, w) < 0$
      - **Send** $u_f(v, w)$ **units of flow on** (v, w)**; update** f**,** e
  - $S(\Delta) = \{v \in V : e(v) \geq \Delta\}$
  - $T(\Delta) = \{v \in V : e(v) \leq -\Delta\}$
  - **while** $S(\Delta) \neq 0$ **and** $T(\Delta) \neq 0$
    - **Pick a node** $k \in S(\Delta)$ **and** $\ell \in T(\Delta)$
    - **Compute** $d(v)$**, shortest path distances from** k **in** $G_f(\Delta)$ **w.r.t. edge distances** c^π**.**
    - **Let** P **be a shortest path from** k **to** ℓ**.**
    - **Set** $\pi = \pi - d$
    - **Let** $\delta = \min\{e(k), -e(\ell), \min\{u_f(v, w) : (v, w) \in P\}\}$
    - **Send** δ **units of flow on the path** P
    - **Update** f**,** $G_f(\Delta)$**,** $S(\Delta)$**,** $T(\Delta)$ **and** c^π**.**
  - $\Delta = \Delta/2$

Analysis of Running Time