Price Competition with the Attraction Demand Model: Existence of Unique Equilibrium and Its Stability

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We show the existence of Nash equilibria in a Bertrand oligopoly price competition game using a possibly asymmetric attraction demand model with convex costs under mild assumptions. We show that the equilibrium is unique and globally stable. To our knowledge, this is the first paper to show the existence of a unique equilibrium with both non-linear demand and non-linear costs. In addition, we guarantee the linear convergence rate of tatônnement. We illustrate the applicability of this approach with several examples arising from operational considerations that are often ignored in the economics literature.

Keywords: price competition; oligopoly; attraction demand model; Nash equilibrium; convex costs.

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1. Introduction

We consider the problem of price equilibrium in markets where multiple firms produce differentiated products. Many firms face the problem of determining pricing and inventory policies in markets where demand depends not only of their own price but also on the prices of competing products. Section 3 lists several examples of operations management problems that fit this framework. Key questions include the existence, uniqueness, and stability of equilibrium when firms are individually setting optimal prices. We present a set of conditions under which a unique and stable pure-strategy equilibrium is guaranteed to exist in a Bertrand oligopoly price competition model when demand is determined by an attraction model and cost functions are convex. We also show that if individual firms follow a "best response approach," to current prices set by other players, then prices converge to the

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Table 1: Common Demand Models.

Demand Model	$d_i(\mathbf{p})$	Restriction
Linear	$(a_i - b_i p_i + c_i \sum_{j \neq i} p_j)^+$	$a_i > 0,\ b_i > 0,\ c_i \geq 0$
Constant Elasticity	$p_i^{1/(\beta-1)} / \left(\sum_j p_j^{\beta/(\beta-1)} \right)^{(1-\theta)/(1-\beta\theta)}$	$\theta, \beta \in (0,1)$
Constant Expenditure	$p_i^{-1} \cdot g(p_i)/\sum_j g(p_j)$	Positive, smooth, and strictly decreasing g †
Logit	$k_i \exp(\lambda p_i)/\sum_j k_j \exp(\lambda p_j)$	$\lambda < 0,\ k_j > 0$
Attraction	$a_i(p_i)/\left(\sum_j a_j(p_j) + \kappa \right)$	$\kappa \geq 0$, Assumption A

† Examples include the CES function $g(p) = p^{-r}$ where $r > 0$, and the exponential function $g(p) = \exp(-\beta p)$ where $\beta > 0$.

unique and stable pure-strategy equilibrium. Finally, we provide the convergence rate of this tatônnement scheme.

Demand and Cost Models

In the Bertrand oligopoly price competition model for differentiated products, a variety of demand and cost models have been used. Table 1 illustrates commonly used demand models. We let n be the number of firms, which are indexed by $i = 1, \ldots, n$. The demand for each firm is specified as a function of prices. Let p_i denote the price of firm i, and define the price vector of competing firms by $\mathbf{p}_{-i} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$. Also denote the vector of all prices by $\mathbf{p} = (p_1, \ldots, p_n) = (p_i, \mathbf{p}_{-i})$. The demand for each firm i is given by $d_i = d_i(\mathbf{p})$. Demand functions are deterministic, but can be interpreted as expected demands in many applications. We assume that firm i's demand is strictly decreasing in its price (i.e., $\partial d_i/\partial p_i < 0$), and that products are gross substitutes (i.e., $\partial d_i/\partial p_j \geq 0$ whenever $j \neq i$).

In this paper we consider a generalization of the logit demand model called the *attraction demand model*:

$$d_i(\mathbf{p}) \ := \ \frac{a_i(p_i)}{\sum_j a_j(p_j) + \kappa} \tag{1}$$

where κ is either 0 or strictly positive.

The *attraction function* $a_i(\cdot)$ of firm i is a positive and strictly decreasing function of its price. Without any loss of generality, we normalize demand such that the total demand does

not exceed 1. If $\kappa = 0$, the total demand of n firms equals exactly 1; if κ is strictly positive, it is less than 1, possibly accounting for lost demand to an outside alternative.

Luce (1959) has shown that the attraction demand model (1) can be derived axiomatically based on simple assumptions about consumer behavior.[2] As discussed in Anderson et al. (1996) and Mahajan and Van Ryzin (1998), the attraction demand model has successfully been used in estimating demand in econometric studies, and is increasingly accepted in marketing, e.g., Besanko et al. (1998). See So (2000), Bernstein and Federgruen (2004b) and references therein for its applications in the operations management community.

We now consider the cost model. We assume that cost is a function of demand alone. We denote firm i's cost function by $C_i(d_i)$ defined on $d_i \in [0, 1]$ and assume C_i is increasing and convex.

The profit of firm i is the difference between its revenue and cost, given by

$$\pi_i \; := \; \pi_i(\mathbf{p}) \; := \; p_i \cdot d_i(\mathbf{p}) - C_i(d_i(\mathbf{p})). \tag{2}$$

Each firm's objective is to maximize π_i.

In this paper, we impose mild technical conditions on the attraction demand and cost models as outlined in Section 2. We then verify that these conditions are satisfied by commonly used attraction functions and cost models.

Literature Review

The study of oligopolistic interaction is a classical problem in economics. In the model proposed by Cournot (1838), firms compete on production output quantities, which in turn determine the market price. In Bertrand (1883)'s model, competition is based on prices instead of production quantities. In the price competition models by Edgeworth (1922) and Edgeworth (1925), each firm decides how much of its demand is satisfied, in which case a pure strategy equilibrium may or may not exist. In addition, price competition with product differentiation has been studied by Hotelling (1929), Robinson (1933) and Chamberlin (1933). An extensive treatment of the subject is found in Vives (1999). We provide a summary of results regarding the existence, uniqueness and stability of equilibrium, followed by their application in the operations management literature.

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[2]Interested readers are referred to Debreu (1960) and Mahajan and Van Ryzin (1998) for discussions on a paradox resulting from Luce (1959)'s axioms. To avoid this paradox, the alternatives in the attraction model should be "equally dissimilar".

Existence. There are two common methods to show existence of an equilibrium in price competition games. The first method is to obtain existence through the quasi-concavity and continuity of π_i in p_i. Assuming a linear cost model, Caplin and Nalebuff (1991) show that a sufficient condition for π_i to be quasi-concave is the concavity of $\log d_i(\mathbf{p})$ in $\log p_i$, which is equivalent to $(p_i/d_i)(\partial d_i/\partial p_i)$ decreasing in p_i. Another sufficient condition is the convexity of $1/d_i(\mathbf{p})$ in p_i. From these conditions, the quasi-concavity of π_i in p_i holds in the logit demand model and the constant expenditure demand model with CES or exponential functions.

The second method shows existence through supermodular games. The price competition game is *supermodular* provided that for each i, π_i is upper semicontinuous in p_i, and $\pi_i(p_i, \mathbf{p}^1_{-i}) - \pi_i(p_i, \mathbf{p}^2_{-i})$ is increasing in p_i whenever $\mathbf{p}^1_{-i} \geq \mathbf{p}^2_{-i}$. Topkis (1979) shows the existence of an equilibrium in supermodular games, and Milgrom and Roberts (1990), for monotone transformation of supermodular games. Thus, if the price competition game is supermodular, it has at least one equilibrium. Similarly, Milgrom and Shannon (1994) show the existence of a Nash equilibrium for a generalization of supermodular games, called games with strategic complementarities. Such games include instances of price competition.

Vives (1999) summarizes two additional methods of showing existence in some special cases: (i) two-player games with decreasing best response functions; (ii) symmetric games in which the best response function of a player depends only on the aggregate actions of others and the action sets are single-dimensional. Dubey et al. (2003) use yet another approach of pseudo-potential games, and show existence and the convergence of a method based on fictitious plays[3]. Yet, they show neither the uniqueness of an equilibrium nor the convergence rate.

Uniqueness. The most common method to show uniqueness is the following contraction condition (see, for example, Milgrom and Roberts (1990)):

$$\frac{\partial^2 \pi_i}{\partial p_i^2} \;+\; \sum_{j \neq i} \left| \frac{\partial^2 \pi_i}{\partial p_i \partial p_j} \right| \;\; < \;\; 0, \tag{3}$$

or a similar condition in which π_i is replaced by $\log \pi_i$. In general, it is not easy to verify this condition on the entire action space unless the demand model is symmetric. An exception is Bernstein and Federgruen (2004b), who show the uniqueness using the attraction demand model and linear cost model under certain conditions. In supermodular price competition

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[3]A fictitious play refers to the strategy in which each player uses best responses to the historical averages of opponents.

games, the contraction condition (3) is satisfied by (i) the linear demand model with convex costs, or (ii) the linear cost model and the constant elasticity, logit or constant expenditure demand models. Thus, these sufficient conditions require either the demand model or the cost model to be linear. In fact, when the action space is unbounded, the above contraction condition (3) is always violated for some large p_i.[4]

Stability. By definition, a set of actions at equilibrium is a fixed point of the best response mapping. A simultaneous discrete *tatônnement* is a sequence of actions in which the current action of each firm is the best response to the previous actions of other firms. An equilibrium is *globally stable* if the tatônnement converges to this equilibrium starting from any initial set of actions. Vives (1990) shows that if a supermodular game with continuous payoffs has a unique equilibrium, it is globally stable. Little is known regarding the provable convergence rate of the tatônnement in the price competition game.

Operations Management Applications. There is a growing interest in oligopolistic price competition in the operations literature. To predict and study market outcomes, the existence and the uniqueness of equilibrium are often required. Stability and convergence rate indicate both the robustness of equilibrium and the efficiency of computational algorithms.

Bernstein and Federgruen (2004b) study a multiple-period inventory model with linear costs where competition is based on both price and service level. They also consider the single-period price-only competition, where the vector of service levels is given exogenously, and the price vector is simultaneously determined by all sellers. This single-period competition is further described in Example 3, of Section 3, where demand uncertainty is multiplicative and there is neither a minimum stocking level nor a capacity constraint(i.e., $\zeta_i = -\infty$ and $v_i = \infty$). For a variety of demand models, including the attraction model and the linear model, an equilibrium exists and is unique. Bernstein and Federgruen (2004a) study comparative statics in pricing competition. Bernstein and Federgruen (2003) and Bernstein et al. (2002) study a supply chain in which multiple competing retailers are replenished from the single firm. In particular, Bernstein et al. (2002) show that the Vendor Managed Inventory arrangement can coordinate the retailers' pricing decision. All of the above models use supermodular games. Price competition in the queueing-based service systems is studied in Allon and Federgruen (2004). In their model, firms compete on price and waiting time. Their paper employs the results of this paper for the attraction demand model with convex

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[4]For any fixed $\mathbf{p}_{-i}$, consider π_i as we increase p_i. Since π_i is strictly decreasing for large p_i (see Proposition 2), and is bounded below (from (2)), the second derivative of π_i must be positive at some point.

costs.

Cachon and Harker (2002) study duopoly price competition with cost function $C_i(d_i) = c_i^1 d_i + c_i^2 d_i^{\gamma_i}$, where $c_i^1 > 0$, $c_i^2 \geq 0$ and $\gamma_i \in [0, 1)$. Note that the cost function is concave in demand. Using the linear demand model or truncated logit demand, they show the existence of equilibrium under certain technical conditions. Uniqueness is shown only for the symmetric case.

Supermodular Games and Price Competition. The existence, uniqueness and stability of equilibrium results are easily obtained in the case of supermodular games. Yet, Bernstein and Federgruen (2004b) suggest that when the game fails to be supermodular, "little can be said ... in general, about the structure or cardinality of the set of Nash equilibria." Vives (1999) shows that with nonsupermodular oligopoly games, "in general, a wide array of outcomes between the monopoly and the competitive solution are possible." Consequently, it is not surprising to note that nearly all uniqueness proofs in price competition rely on the supermodularity of games and an imposed contraction assumption. Yet, Vives (1999) remarks that oligopoly price competition games need not be supermodular, and points out examples by Roberts and Sonnenschein (1977) and Friedman (1983). In the price competition literature, if a supermodular game has an explicit demand model and cost model, it typically assumes either the linear demand model or the linear cost model. The only exceptions are Cachon and Harker (2002), Milgrom and Shannon (1994), and Mizuno (2003). The first is restricted to symmetric duopoly. The second shows the existence but not the uniqueness. It does not require a convex action space, but requires a compact action space. The third shows the uniqueness under a strong assumption on demand: namely, the demand vector remains the same if the price vector is increased uniformly or multiplied by a scalar.

Contribution and Organization

We show the uniqueness of equilibrium in Bertrand oligopoly price competition using an attraction demand model with convex costs. We illustrate the applicability of our generalization by illustrating convex cost models that arise in inventory and service systems.

Second, the action space of most previous Bertrand oligopoly models is assumed to be compact. Thus, even if existence can be shown for the compact space, an equilibrium is possibly a boundary solution, and no interior equilibrium may exist within the compact action space. Consequently, to identify an interior equilibrium solution, an additional assumption needs to be introduced as in Benassy (1989) and Vives (1985). In this paper, we identify

sufficient conditions for the unique equilibrium to be in the interior of the set. In particular, we allow unbounded action spaces.

Finally, we show that the unique equilibrium is globally stable and guarantee a linear convergence rate of tatônnement regardless of the initial actions. This is the first paper that identifies the convergence rate of simultaneous discrete tatônnement in a Bertrand oligopoly price competition game. Since the unique equilibrium generally does not have an analytic solution, this result is useful in the numerical computation of the equilibrium.

The organization of the rest of this paper is as follows. Section 2 outlines our modeling assumptions on the attraction functions and the cost functions. This section also shows that many common attraction functions satisfy these assumptions. Section 3 lists convex cost examples in operations management, to which our model becomes applicable. Section 4 proves the existence of unique equilibrium, also shown to be globally stable in Section 5. Section 5 also proves the linear convergence of the tatônnement scheme. Computational results follow in Section 6.

2. Assumptions

This section lists our assumptions on the attraction function $a_i(\cdot)$ in (1), the profit function π_i and the cost function $C_i(\cdot)$ in (2). We show that these assumptions are satisfied by common attraction functions.

We let $\rho_i := \inf\{p : a_i(p) = 0\}$ be the upper bound on price p_i, where ρ_i may be infinite. Firm i's action space for price is an open interval $(0, \rho_i)$. Let $\mathcal{P} := (0, \rho_1) \times \cdots \times (0, \rho_n)$. Let

$$\eta_i(p) := -p \cdot a_i'(p)/a_i(p)$$

be the *elasticity* of firm i's attraction function. We adopt the following simplifying notation: $f(x+) := \lim_{h \downarrow x} f(h)$, $f(x-) := \lim_{h \uparrow x} f(h)$, $\inf \emptyset = \infty$, and $\frac{y}{y+k} = 1$ if $y = \infty$ and k is finite.

Condition A. For each firm i,

(A1) $a_i(\cdot)$ is positive, strictly decreasing and continuously differentiable, i.e., $a_i(p) > 0$ and $a_i'(p) < 0$ for all $p \in (0, \rho_i)$. It follows that the elasticity of attraction, $\eta_i(\cdot)$, is positive for $p \in (0, \rho_i)$.

(A2) The elasticity $\eta_i(\cdot)$ is nondecreasing. In other words, when the price is higher, the percentage decrease in the attraction function per percentage change in price is also

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higher. This assumption is central to the existence and uniqueness proofs. It is used, for example, to show that the first order condition (4) has a unique solution p_i for any $\mathbf{p}_{-i}$.

(A3) If $a_i(0+) < \infty$, then $a_i'(0+) > -\infty$. It follows that when the price is low enough, the elasticity of attraction is close to 0, i.e., $\eta_i(0+) = 0$. This is needed to prove the interiority of equilibria (Proposition 3); in any equilibrium, firm i would never set its price $p_i = 0$. We remark that (A3) holds vacuously if $a_i(0+) = \infty$.

Condition B. For each firm i,

(B1) $C_i(\cdot)$ is strictly increasing, continuously differentiable, and convex on $[0, 1]$ (i.e., $c_i(\cdot) := C_i'(\cdot)$ is positive and increasing), and satisfies $c_i(0+) > 0$.

Condition C. For each firm i,

(C1) $c_i(0) < \rho_i \cdot (1 - 1/\eta_i(\rho_i))$, i.e., the Lerner index $[p_i - c_i(d_i)]/p_i$ at price $p_i = \rho_i$ and demand $d_i = 0$ is strictly larger than $1/\eta_i(\rho_i)$. It means that no firm would set the maximum price in equilibrium because its profits are decreasing at this price. It is needed for the interiority of equilibria.

(C2) If $\kappa = 0$, then $c_i(1) < \rho_i$. It means that firm i could profitably serve all demand.

(C3) If $\kappa = 0$, the following technical condition holds:

$$\sum_{i=1}^{n} \left(1 - \frac{1}{\eta_i(\rho_i) \cdot (1 - c_i(1)/\rho_i)} \right) > 1 .$$

This inequality holds when there are at least two distinct firms i satisfying $\eta_i(\rho_i) = \infty$. Note $\rho_i < \infty$ implies $\eta_i(\rho_i) = \infty$.

We note that both (C2) and (C3) are used for the uniqueness proof only (Proposition 6).

Proposition 1 shows that these conditions are satisfied for arbitrary cost functions satisfying Conditions (B1) and (C2):

- Generalized Logit Attraction Function: $a_i(p) = \exp(-\lambda_i p)$ with $\lambda_i > 0$

- Cobb-Douglas Attraction Function: $a_i(p) = p^{-\gamma_i}$ with $\gamma_i > 1$

- Linear Attraction Function: $a_i(p) = \alpha_i - \beta_i p$ with $\alpha_i, \beta_i > 0$.

Proposition 1 *Suppose $n \geq 2$, and cost function $C_i(\cdot)$ satisfies Conditions (B1) and (C2) for each i. Then, Conditions A, B and C are satisfied provided that each firm i's attraction function is either (i) generalized logit, (ii) Cobb-Douglas with an additional assumption $\gamma_i > 2$ in case of $\kappa = 0$, or (iii) linear where $\alpha_i/\beta_i > c_i(1)$.*

Proof: (i) We get $\rho_i = \infty$ and $a_i'(p) = -\lambda_i \exp(-\lambda_i p)$. Thus, $\eta_i(p) = \lambda_i p$, $\eta_i(\rho_i) = \infty$, $a_i(0+) = 1$, and $a_i'(0+) = -\lambda_i$, satisfying Condition A. In addition, since $c_i(0)$ is finite, $\rho_i \cdot (1 - 1/\eta_i(\rho_i)) = \infty$ implies (C1). Furthermore, $1 - (\eta_i(\rho_i) \cdot (1 - c_i(1)/\rho_i))^{-1} = 1$.

(ii) We have $\rho_i = \infty$ and $a_i'(p) = -\gamma_i p^{-\gamma_i - 1}$. Thus, the elasticity $\eta_i(p) = \gamma_i$ is constant for all $p \in (0, \rho_i)$, and $a_i(0+) = \infty$, satisfying Condition A. Also, (C1) follows from $\rho_i \cdot (1 - 1/\eta_i(\rho_i)) = \rho_i \cdot (1 - 1/\gamma_i) = \infty$. If $\kappa = 0$, then we assume $\gamma_i > 2$ and get $1 - (\eta_i(\rho_i) \cdot (1 - c_i(1)/\rho_i))^{-1} = 1 - 1/\gamma_i > 1/2$.

(iii) We have $\rho_i = \alpha_i/\beta_i$ and $a_i'(p) = -\beta_i$. Thus, $\eta_i(p) = \beta_i p/(\alpha_i - \beta_i p)$, and $a_i(0+) = \alpha_i$, satisfying Condition A. In addition, we obtain $\eta_i(\rho_i) = \infty$, and thus $\rho_i \cdot (1 - 1/\eta_i(\rho_i)) = \rho_i = \alpha_i/\beta_i > c_i(1) \geq c_i(0)$, implying (C1). Since $1 - c_i(1)/\rho_i > 0$, we obtain $1 - (\eta_i(\rho_i) \cdot (1 - c_i(1)/\rho_i))^{-1} = 1$.

Now, (C3) follows easily since in each of the above three cases, since $1 - (\eta_i(\rho_i) \cdot (1 - c_i(1)/\rho_i))^{-1}$ is strictly greater than $1/2$. ∎

Condition (B1) essentially means that $C_i(\cdot)$ is a smooth convex function in d_i. Examples of $C_i(\cdot)$ include the linear function and exponential function. More examples are provided in Section 3. We remark that attraction functions do not need to be identical. Furthermore, even the form of the attraction function may not be same among firms. Analogously, the cost functions need not have the same form either.

For the rest of this paper, we assume Conditions A, B and C hold. In Section 5, we introduce as an additional assumption that both $C_i(\cdot)$ and $a_i(\cdot)$ are twice continuously differentiable.

3.   Examples

In this section, we list price competition models for which the convex cost model is applicable. With the attraction demand model, the results of this paper show the existence and uniqueness of equilibrium in these models. We present some examples from inventory-capacity systems, followed by those from service systems based on queues.

Inventory-Capacity Systems

EXAMPLE 1. Consider the pricing problem in the stochastic inventory system with exogenously determined stocking levels. We denote stochastic demand of firm i by $D_i(\mathbf{p})$, and its expected demand by $d_i(\mathbf{p})$. Demand is a function of the price vector $\mathbf{p} = (p_1, \ldots, p_n)$. We represent firm i's stochastic demand by $D_i(p_1, \ldots, p_n) = \varphi(d_i(\mathbf{p}), \varepsilon_i)$, where ε_i is a random variable. (We can allow φ to be dependent on i). We suppose the continuous density function $f_i(\cdot)$ for ε_i exists, and let $F_i(\cdot)$ denote its cumulative density function.

Let y_i be the exogenously fixed stocking level of firm i. For the first y_i units, the per-unit materials cost is w_i. If realized demand is at most y_i, the per-unit salvage value of $w_i - h_i > 0$ is obtained. Otherwise, the excess demand is met through an emergency supply at the cost of $w_i + b_i$ per unit, where $b_i \geq 0$. The profit of firm i is the difference between its revenue and costs, and the expected profit is $\pi_i(\mathbf{p}|y_i) = p_i \cdot d_i(\mathbf{p}) - C_i(d_i(\mathbf{p}), y_i)$, where

$$C_i(d_i, y_i) = w_i d_i + h_i E[y_i - \varphi(d_i, \varepsilon_i)]^+ + b_i E[\varphi(d_i, \varepsilon_i) - y_i]^+ \ ,$$

and h_i and b_i are the per-unit inventory overage and underage costs, respectively.

Our goal is to show that for fixed y_i, this function satisfies condition (B1). We achieve this goal with two common demand uncertainty models.

- Additive Demand Uncertainty Model: $\varphi(d_i, \varepsilon_i) = d_i + \varepsilon_i$ where $E[\varepsilon_i] = 0$. Then,

$$\begin{aligned}
\frac{\partial C_i(d_i, y_i)}{\partial d_i} &= w_i - h_i P[y_i \geq d_i + \varepsilon_i] + b_i P[y_i \leq d_i + \varepsilon_i] \\
&= w_i - h_i F_i(y_i - d_i) + b_i(1 - F_i(y_i - d_i)).
\end{aligned}$$

- Multiplicative Demand Uncertainty Model: $\varphi(d_i, \varepsilon_i) = d_i \cdot \varepsilon_i$ where ε_i is positive and $E[\varepsilon_i] = 1$. Then,

$$\begin{aligned}
\frac{\partial C_i(d_i, y_i)}{\partial d_i} &= w_i - h_i \int_0^{y_i/d_i} \varepsilon dF_i(\varepsilon) + b_i \int_{y_i/d_i}^{\infty} \varepsilon dF_i(\varepsilon) \\
&= w_i - h_i + (h_i + b_i) \int_{y_i/d_i}^{\infty} \varepsilon dF_i(\varepsilon).
\end{aligned}$$

In both cases, $\partial C_i(d_i, y_i)/\partial d_i$ is positive since $w_i > h_i$ and nondecreasing in d_i. We conclude that for fixed y_i, $C_i(d_i, y_i)$ is strictly increasing, twice continuously differentiable, and convex in d_i. Furthermore, $\partial C_i(d_i, y_i)/\partial d_i > 0$ at $d_i = 0$.

EXAMPLE 2. We modify Example 1 to allow the stocking decision to exist as an operational decision, where the stocking level is constrained. The sequence of events occurs as follows. (1) Firms decide prices simultaneously. (2) The price vector $\mathbf{p} = (p_1, \ldots, p_n)$ is announced. (3) Each firm i decides on the stocking level y_i, subject to the minimum stocking level and capacity constraint $\zeta_i \leq y_i \leq \upsilon_i$. (4) For each firm i, demand is realized, and the appropriate overage and underage cost is incurred.

For any given $\mathbf{p}$, firm i's expected demand is given by $d_i = d_i(\mathbf{p})$, and its optimal stocking level can be computed. Let $y_i^{NV}(d_i)$ be the solution to the newsvendor problem, i.e., $P[\varphi(d_i, \varepsilon_i) \leq y_i^{NV}(d_i)] = \frac{b_i}{b_i + h_i}$ where b_i and h_i are defined in Example 1. Let $y_i^*(d_i)$ be the optimal stocking level, which is y_i maximizing $\pi_i(\mathbf{p}|y_i)$ subject to $\zeta_i \leq y_i \leq \upsilon_i$. Then, $y_i^*(d_i)$ is the point in the interval $[\zeta_i, \upsilon_i]$ that is the closest to $y_i^{NV}(d_i)$, i.e., $y_i^*(d_i) = \max\{\zeta_i, \min\{y_i^{NV}(d_i), \upsilon_i\}\}$.

In the additive demand uncertainty model, we get $y_i^*(d_i) = \max\{\zeta_i, \min\{d_i + z_i, \upsilon_i\}\}$, where $z_i = F_i^{-1}(b_i/(b_i + h_i))$. Then,

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\begin{aligned}
C_i(d_i, y_i^*(d_i)) &= w_i d_i + h_i E[y_i^*(d_i) - \varphi(d_i, \varepsilon_i)]^+ + b_i E[\varphi(d_i, \varepsilon_i) - y_i^*(d_i)]^+ \\
&= \begin{cases} w_i d_i + h_i E[\zeta_i - d_i - \varepsilon_i]^+ + b_i E[d_i + \varepsilon_i - \zeta_i]^+, & \text{if } \zeta_i > d_i + z_i \text{ or } \zeta_i > \upsilon_i \\ w_i d_i + h_i E[z_i - \varepsilon_i]^+ + b_i E[\varepsilon_i - z_i]^+, & \text{if } \zeta_i \leq d_i + z_i \leq \upsilon_i \\ w_i d_i + h_i E[\upsilon_i - d_i - \varepsilon_i]^+ + b_i E[d_i + \varepsilon_i - \upsilon_i]^+, & \text{if } d_i + z_i > \upsilon_i \text{ and } \zeta_i \leq \upsilon_i. \end{cases}
\end{aligned}
$$

We can easily show that $C_i(d_i, y_i^*(d_i))$ is strictly increasing, twice differentiable and convex in d_i. Similarly, we obtain the analogous result for the multiplicative demand uncertainty model, in which $y_i^*(d_i) = \min\{d_i z_i, \upsilon_i\}$. Furthermore, if $\zeta_i = -\infty$ and $\upsilon_i = \infty$, then $C_i(d_i, y_i^*(d_i))$ is linear in d_i.

EXAMPLE 3. We modify Example 2 to model the minimum service level requirement. Suppose that a vector $(f_1, \cdots, f_n)$ is exogenously specified such that firm i's stock-out probability should be at most $1 - f_i$. We remove the capacity constraint (i.e., $\upsilon_i = \infty$ for all i) to avoid infeasibility. Then, all the results in Example 2 continue to hold, with the redefinition of $z_i = F_i^{-1}(\max\{f_i, b_i/(b_i + h_i)\})$.

Consider a special case of this example in which the stocking levels are determined solely

by the $(f_1, \cdots, f_n)$ vector (i.e., $z_i = F_i^{-1}(f_i)$), and the minimum stocking levels do not exist (i.e., $\zeta_i = -\infty$). Then, C_i is linear. In this case, the uniqueness and existence are first proven by Bernstein and Federgruen (2004b) using supermodular games.

We remark that in Examples 2 and 3, the second derivative of C_i with respect to d_i may not be continuous. In such cases, the existence and uniqueness still hold; yet, our results regarding stability and convergence rate in Section 5 do not apply.

Service Systems

EXAMPLE 4. We model each firm as a single server queue with finite buffer, where the firms' buffer sizes are given exogenously. Let κ_i denote the size of firm i's buffer; no more than κ_i customers are allowed to the system. We assume exponential service times and the Poisson arrival process. The rate μ_i of service times are exogenously determined, and the rate d_i of Poisson arrival is an output of the price competition. In the queueing theory notation, each firm i is a $M/M/1/\kappa_i$ system.

We assume that the materials cost is $w_i > 0$ per served customer, and the diverted customers' demand due to buffer overflow is met by an emergence supply at the cost of $w_i + b_i$ unit per customer, where $b_i > 0$. The demand arrival rate $d_i = d_i(\mathbf{p})$ is determined as a function of the price vector $\mathbf{p}$. It follows that firm i's time-average revenue is $p_i \cdot d_i - C_i(d_i)$, where $C_i(d_i)$ is the sum of $w_i \cdot d_i$ and the time-average number of customers diverted from the system is multiplied by b_i. Thus, according to elementary queueing theory (see, for example, Medhi (2003)),

$$
\begin{aligned}
C_i(d_i) &= w_i \cdot d_i + b_i \cdot \frac{d_i \cdot (1 - d_i/\mu_i)(d_i/\mu_i)^{\kappa_i}}{1 - (d_i/\mu_i)^{\kappa_i+1}}, \quad \text{if } d_i \neq \mu_i \\
&= w_i \cdot d_i + b_i \cdot \frac{d_i}{\kappa_i + 1}, \quad \text{if } d_i = \mu_i.
\end{aligned}
$$

Algebraic manipulation shows that $C_i(\cdot)$ is convex and continuously twice differentiable, satisfying $c_i(0) = w_i > 0$.

EXAMPLE 5. Consider price competition among $M/D/1$ queueing systems in which the service level θ_i of each firm i is exogenously specified. The following sequence of events occurs. (1) Firms decide their prices simultaneously. (2) The price vector $\mathbf{p} = (p_1, \ldots, p_n)$ is announced. (3) Each firm i decides on its capacity μ_i, which is the service rate. It

takes exactly $1/\mu_i$ time units to serve a customer. We assume that firm i's capacity cost is proportional to its capacity. The service level is defined as the reciprocal of the expected amount of time spent in the system, and capacity should be sufficient enough to satisfy the minimum service level. (4) For each firm, customers arrive at each queueing system according to a Poisson process with rate d_i. We assume zero marginal cost of production.

The expected amount of time spent by a customer in firm i's queueing system is given by

$$\frac{1}{\theta_i} = \frac{1}{\mu_i} + \frac{d_i}{2(\mu_i - d_i)\mu_i}.$$

Thus, using the quadratic formula, firm i's capacity should be

$$\mu_i = \frac{d_i}{2} + \frac{\theta_i}{2} + \frac{\sqrt{d_i^2 + \theta_i^2}}{2},$$

which is convex in d_i. Recall that firm i's expected profit $p_i \cdot d_i - C_i(d_i)$, where $C_i(d_i)$ is proportional to μ_i. The cost function C_i is convex and continuously twice differentiable. Furthermore $C_i'(0) > 0$.

We remark that this simple example is a special case of a general model presented in Allon and Federgruen (2004). They provide extensive treatment of $G/G/1$ queueing systems.

4.   Existence and Uniqueness of Equilibrium

In this section, we show that the oligopoly price competition has a unique equilibrium. Given the price vector, each firm's profit function is given by expression (2), where its demand is determined by (1). We first show that the first order condition $\partial \pi_i / \partial p_i = 0$ is sufficient for the Nash equilibrium (Proposition 2). For each value of a suitably defined aggregate attraction δ, we show that there is at most one candidate for the solution of the first order condition (Proposition 3). Then, we demonstrate that there exists a unique value δ of the aggregator such that this candidate indeed solves the first order condition (Propositions 5 and 6). We proceed by assuming both Conditions A, B and C.

Let $\varsigma_i(\mathbf{p}) := \eta_i(p_i) \cdot (1 - d_i(\mathbf{p}))$.

Proposition 2 *Firm i's profit function π_i is strictly quasi-concave in $p_i \in (0, \rho_i)$. A vector of prices $\mathbf{p}^* = (p_1^*, \ldots, p_n^*) \in \mathcal{P}$ satisfies $\partial \pi_i(\mathbf{p}^*)/\partial p_i = 0$ for all i if and only if $\mathbf{p}^*$ is a Nash equilibrium in $\mathcal{P}$. Also, $p_i^* > c_i(0)$ for each i. Furthermore, the condition $\partial \pi_i / \partial p_i = 0$ is*

equivalent to

$$\frac{c_i(d_i(\mathbf{p}))}{p_i} = 1 - \frac{1}{\varsigma_i(\mathbf{p})} . \tag{4}$$

Note that the condition in (4) is analogous to the inverse elasticity condition for optimal monopoly pricing.

Proof: Compute the partial derivative of π_i with respect to $p_i \in (0, \rho_i)$. We abuse the prime notation and use it for the partial differentiation with respect to p_i.

$$
\begin{aligned}
\frac{\partial \pi_i}{\partial p_i} &= d_i' p_i + d_i - C_i'(d_i) d_i' \\
&= \left(\frac{a_i}{\sum a_j + \kappa} \right)' (p_i - c_i(d_i)) + \frac{a_i}{\sum a_j + \kappa} \\
&= \frac{a_i'(\sum a_j + \kappa) - a_i a_i'}{(\sum a_j + \kappa)^2} (p_i - c_i(d_i)) + \frac{a_i}{\sum a_j + \kappa} \\
&= \frac{a_i'(1 - d_i)}{\sum a_j + \kappa} (p_i - c_i(d_i)) + \frac{a_i}{\sum a_j + \kappa} \\
&= \left(\frac{(1 - d_i) p_i a_i'}{\sum a_j + \kappa} \right) \left(\frac{p_i - c_i(d_i)}{p_i} + \frac{a_i}{p_i a_i'} \frac{1}{1 - d_i} \right) .
\end{aligned}
\tag{5}
$$

Note that the first factor is negative. Thus, $\partial \pi_i / \partial p_i = 0$ is equivalent to setting the second factor to zero, which yields (4).

Let $l_i(p_i)$ and $r_i(p_i)$ be the left and right sides of (4) respectively, i.e.,

$$l_i(p_i) := \frac{c_i(d_i(\mathbf{p}))}{p_i} \quad \text{and} \quad r_i(p_i) := 1 - \frac{1}{\varsigma_i(\mathbf{p})}.$$

We claim that $r_i(p_i) - l_i(p_i)$ is a strictly increasing function of p_i. To see this, since $d_i(\mathbf{p})$ is strictly decreasing in p_i, by (B1), $l_i(\cdot)$ is strictly decreasing. Also, since $1 - d_i(\mathbf{p})$ is positive and strictly increasing in p_i, (A2) implies that $\varsigma_i(\mathbf{p})$ is positive and strictly increasing in p_i. Hence, $r_i(\cdot)$ is strictly increasing. Thus, the claim follows.

The first factor in (5) is negative, and the second factor is $r_i(p_i) - l_i(p_i)$, which increases strictly. Thus, $\partial \pi_i / \partial p_i$ has at most one sign change (from positive to negative), and the change occurs when $l_i(p_i) = r_i(p_i)$, or $\partial \pi_i / \partial p_i = 0$. It follows that π_i is strictly quasi-concave in p_i. By quasi-concavity, $\mathbf{p}^* = (p_1^*, \ldots, p_n^*) \in \mathcal{P}$ is a Nash equilibrium if it satisfies $\frac{\partial}{\partial p_i} \pi_i(\mathbf{p}^*) = 0$ for all i. Since $\mathbf{p}^*$ is an interior solution, $\frac{\partial}{\partial p_i} \pi_i(\mathbf{p}^*) \neq 0$ for some i implies that $\mathbf{p}^*$ is not a Nash equilibrium.

Furthermore, since $\varsigma_i(\mathbf{p}^*)$ is positive, $(c_i(d_i(\mathbf{p}^*)) - p_i^*)/p_i^*$ is negative by (4). Thus, $p_i^* > c_i(d_i(\mathbf{p}^*)) \geq c_i(0)$, where the second inequality follows from (B1). $\blacksquare$

Given a price vector, let $\delta := \sum_{j=1}^{n} a_j(p_j)$ be the aggregate attraction. The support of δ is $\Delta := \left(0, \sum_{j=1}^{n} a_j(0+)\right)$. From (A1), it follows that $\delta \in \Delta$. Then, $d_i = a_i(p_i)/(\delta + \kappa)$. Since a_i^{-1} is well-defined by (A1), we get $p_i = a_i^{-1}((\delta + \kappa)d_i)$. Thus, (4) is equivalent to

$$\frac{c_i(d_i)}{a_i^{-1}((\delta + \kappa)d_i)} = 1 - \frac{1}{\eta_i \circ a_i^{-1}((\delta + \kappa)d_i) \cdot (1 - d_i)}. \tag{6}$$

Observe that there is one-to-one correspondence between $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{d} = (d_1, \ldots, d_n)$, given δ (and of course, κ). Let $D_i(\delta)$ be the solution to (6) given δ (and κ). The existence and uniqueness of $D_i(\delta)$ are guaranteed by Proposition 3 below. The $D_i(\delta)$'s may not sum up to the "correct" value of $\delta/(\delta + \kappa)$ unless a set of conditions is satisfied (Propositions 5). Proposition 6 shows the existence of a unique δ such that the $D_i(\delta)$'s sum up to $\delta/(\delta + \kappa)$.

Let $\overline{d}_i(\delta) := \min\left\{\frac{a_i(0+)}{\delta + \kappa}, 1\right\}$ be an upper bound on the market share of firm i. For each fixed $\delta \in \Delta$, we define the following real-valued functions on $\left(0, \overline{d}_i(\delta)\right)$:

$$L_i(x_i|\delta) := \frac{c_i(x_i)}{a_i^{-1}((\delta + \kappa)x_i)} \quad \text{and} \quad R_i(x_i|\delta) := 1 - \frac{1}{\eta_i \circ a_i^{-1}((\delta + \kappa)x_i)\,(1 - x_i)}. \tag{7}$$

We remark that both $L_i(x_i|\delta)$ and $R_i(x_i|\delta)$ are continuous in x_i in $\left(0, \overline{d}_i(\delta)\right)$.

Proposition 3 *For each i and each $\delta \in \Delta$, $L_i(\cdot|\delta)$ is positive and strictly increasing, and $R_i(\cdot|\delta)$ is strictly decreasing. Furthermore, $L_i(x_i|\delta) = R_i(x_i|\delta)$ has a unique solution in $\left(0, \overline{d}_i(\delta)\right)$, i.e., $D_i(\delta)$ is a well-defined function of δ.*

Proof: Fix δ. Since $a_i^{-1}((\delta + \kappa)x_i)$ is a strictly decreasing function of x_i, (B1) implies that $L_i(x_i|\delta)$ is positive and strictly increasing in x_i. Furthermore, since $\eta_i \circ a_i^{-1}(\cdot)$ is positive and decreasing on $(0, a_i(0+))$, $R_i(x_i|\delta)$ is strictly decreasing in x_i. Thus, $L_i(x_i|\delta) = R_i(x_i|\delta)$ has at most one solution.

¿From (C1), $c_i(0) < \rho_i \cdot (1 - 1/\eta_i(\rho_i))$ implies

$$L_i(0 + |\delta) = \frac{c_i(0)}{\rho_i} < 1 - \frac{1}{\eta_i(\rho_i)} = R_i(0 + |\delta) .$$

Now, as $x_i \to \overline{d}_i(\delta) = \min\left\{\frac{a_i(0+)}{\delta + \kappa}, 1\right\}$, we claim that $R_i(x_i|\delta) \to -\infty$. To see this, consider the following two cases:

- Case $\frac{a_i(0+)}{\delta + \kappa} > 1$: We have $0 < a_i^{-1}(\delta + \kappa) < \rho_i$, which implies $0 < \eta_i \circ a_i^{-1}(\delta + \kappa) < \infty$. Thus, as $x_i \to 1$, we have $1/(1 - x_i) \to \infty$, and $R_i(x_i|\delta) \to -\infty$

- Case $\frac{a_i(0+)}{\delta+\kappa} \leq 1$: Suppose $x_i \to \frac{a_i(0+)}{\delta+\kappa}$. Since $a_i(0+)$ is finite, (A3) implies that $\eta_i \circ a_i^{-1}((\delta+\kappa)x_i)$ approaches 0 from above. Furthermore, $1-x_i$ approaches a nonnegative number from above. Thus, $R_i(x_i|\delta) \to -\infty$.

Recall that $L_i(x_i|\delta)$ is positive. Thus, from the above claim, there exists at least one solution satisfying $L_i(x_i|\delta) = R_i(x_i|\delta)$ by the Intermediate Value Theorem. ∎

For any aggregate attraction $\delta \in \Delta$, Proposition 3 shows that there is a unique solution x_i satisfying $L_i(x_i|\delta) = R_i(x_i|\delta)$, and this solution is $D_i(\delta)$. It represents demand that maximizes firm i's profit provided that the aggregate attraction remains at δ. Also, define

$$D(\delta) \; := \; D_1(\delta) + \cdots + D_n(\delta).$$

Proposition 4 *All $D_i(\delta)$'s and $D(\delta)$ are strictly decreasing functions. Furthermore, each $D_i(\delta)$ is a continuous function of δ.*

Proof: For each i, $D_i(\delta)$ is a strictly decreasing function since any increase in δ lifts the graph of $L_i(x_i|\delta)$ and drops that of $R_i(x_i|\delta)$. For the continuity of D_i, see Appendix A.1. ∎

Proposition 5 *For fixed $\delta \in \Delta$, $D(\delta) = \frac{\delta}{\delta+\kappa}$ holds if and only if there exist $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{d} = (d_1, \ldots, d_n)$ such that the following set of conditions hold:*

(i) $\delta = \sum_{j=1}^{n} a_j(p_j)$,

(ii) $d_i = a_i(p_i)/(\delta + \kappa)$ for each i, and

(iii) $L_i(d_i|\delta) = R_i(d_i|\delta)$ for each i.

In this case, furthermore, the price vector corresponding to any δ satisfying $D(\delta) = \frac{\delta}{\delta+\kappa}$ is unique.

Proof: Suppose there exist $\mathbf{p} \in \mathcal{P}$ and $\mathbf{d}$ satisfying (i), (ii) and (iii). Then, by definition of $D_i(\delta)$ and (iii), we obtain $d_i = D_i(\delta)$, and

$$D(\delta) = \sum_{i=1}^{n} D_i(\delta) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \frac{a_i(p_i)}{\delta + \kappa} = \frac{\delta}{\delta + \kappa}.$$

Conversely, suppose δ satisfies $D(\delta) = \frac{\delta}{\delta+\kappa}$. Set $d_i = D_i(\delta)$. Then, by definition of $D_i(\delta)$, (iii) holds. Let $p_i = a_i^{-1}((\delta + \kappa)d_i)$ for each i, and (ii) holds. Also, (i) follows from

$$\sum_{j=1}^{n} a_j(p_j) = \sum_{j=1}^{n} (\delta + \kappa)d_j = \sum_{j=1}^{n} (\delta + \kappa)D_j(\delta) = (\delta + \kappa)D(\delta) = \delta.$$

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We have established the *if and only if* portion, and now consider the uniqueness portion. For any given δ, the $L_i(d_i|\delta) = R_i(d_i|\delta)$ equation has only one solution by Proposition 3. Hence, from the one-to-one correspondence between $\mathbf{p}$ and $(\delta; \mathbf{d})$ given δ, the uniqueness of d_i implies the uniqueness of the price vector $\mathbf{p}$. ∎

If there is $\delta \in \Delta$ satisfying $D(\delta) = \frac{\delta}{\delta + \kappa}$, then by Proposition 5, the corresponding price vector satisfies $\partial \pi_i / \partial p_i = 0$ for all i. By Proposition 2, this price vector is a Nash equilibrium. For the unique existence of the equilibrium, it suffices to show the result of the following proposition.

Proposition 6 *There exists a unique $\delta \in \Delta$ such that $D(\delta) = \delta/(\delta + \kappa)$.*

Proof: There is at most one $\delta \in \Delta$ such that $D(\delta) = \delta/(\delta + \kappa)$ since $D(\delta) - \delta/(\delta + \kappa)$ is a strictly decreasing function by Proposition 4. Since each D_i is monotone and bounded in $[0,1]$, $\lim_{\delta \downarrow 0} D(\delta)$ in $[0,1]$ and $\lim_{\delta \uparrow \sum_i a_i(0+)} D(\delta) - \delta/(\delta + \kappa)$ exist. We claim (i) $\lim_{\delta \downarrow 0} D(\delta) - \delta/(\delta + \kappa) > 0$ and (ii) $\lim_{\delta \uparrow \sum_i a_i(0+)} D(\delta) - \delta/(\delta + \kappa) < 0$. From the continuity $D(\delta) - \delta/(\delta + \kappa)$, these two claims are sufficient for the result.

First we show (i). Suppose $\kappa = 0$. We note that the domain of both L_i and R_i functions is $\left(0, \overline{d}_i(\delta)\right)$, and

$$D_i(0+) \;\leq\; \lim_{\delta \downarrow 0} \overline{d}_i(\delta) \;=\; \lim_{\delta \downarrow 0} \min\{\frac{a_i(0+)}{\delta}, 1\} \;=\; 1,$$

where the last equality comes from $\lim_{\delta \downarrow 0} a_i(\delta) > 0$. By rewriting equation (6) and the continuity of all involved functions,

$$
\begin{aligned}
D_i(0+) \;&=\; \lim_{\delta \downarrow 0} D_i(\delta) \\
&=\; 1 - \lim_{\delta \downarrow 0} \frac{1}{\eta_i(a_i{}^{-1}(\delta D_i(\delta)))} \cdot \frac{1}{1 - c_i(D_i(\delta))/a_i{}^{-1}(\delta D_i(\delta))} \\
&=\; 1 - \frac{1}{\eta_i(a_i{}^{-1}(0+))} \cdot \frac{1}{1 - c_i(D_i(0+))/a_i{}^{-1}(0+)} \\
&=\; 1 - \frac{1}{\eta_i(\rho_i)} \cdot \frac{1}{1 - c_i(D_i(0+))/\rho_i} \\
&\geq\; 1 - \frac{1}{\eta_i(\rho_i)} \cdot \frac{1}{(1 - c_i(1)/\rho_i)}
\end{aligned}
$$

where the last inequality follows from (B1) and (C2). Then,

$$D(0+) - \lim_{\delta \downarrow 0} \frac{\delta}{\delta + \kappa} \;=\; \sum_i D_i(0+) - 1 \;\geq\; \sum_i \left(1 - \frac{1}{\eta_i(\rho_i)(1 - c_i(1)/\rho_i)}\right) - 1$$

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which is greater than 0 by (C3).

Suppose $\kappa > 0$. We temporarily denote $L_i(\cdot|\delta)$ and $R_i(\cdot|\delta)$ by $L_i(\cdot|\delta, \kappa)$ and $R_i(\cdot|\delta, \kappa)$, respectively. Let $\alpha := \min\{\kappa/2, \sum_i a_i(0+)/2\} > 0$. Then,

$$L_i(d_i|0+, \kappa) = \frac{c_i(d_i)}{a_i^{-1}(\kappa d_i)} = L_i(d_i|\alpha, \kappa - \alpha)$$

$$R_i(d_i|0+, \kappa) = 1 - \frac{1}{\eta_i \circ a_i^{-1}(\kappa d_i)(1 - d_i)} = R_i(d_i|\alpha, \kappa - \alpha).$$

By definition of $D_i(0+)$, we get $L_i(D_i(0+)|\alpha, \kappa - \alpha) = R_i(D_i(0+)|\alpha, \kappa - \alpha)$. By Proposition 3, this condition implies $D_i(0+) > 0$. Furthermore, $\delta/(\delta + \kappa) \to 0$ as $\delta \downarrow 0$. Thus, we complete the proof of claim (i).

Now we show (ii). From the equation (6) and definition (7),

$$(\delta + \kappa)D_i(\delta) = a_i\left(\frac{c_i(D_i(\delta))}{R_i(D_i(\delta)|\delta)}\right) < a_i\left(\frac{c_i(0)}{R_i(0|\delta)}\right) = a_i\left(\frac{c_i(0)}{1 - 1/\eta_i(\rho_i)}\right),$$

where the inequality follows from (B1), $D_i(\delta) > 0$, and Proposition 3. Since $c_i(0)$ is positive from (B1), (C1) implies $1 - 1/\eta_i(\rho_i) > 0$ and $0 < \frac{c_i(0)}{1-1/\eta_i(\rho_i)} < \rho_i$. Since $a_i(\cdot)$ is strictly decreasing by (A1), we get

$$\bar{\delta} := \sum_i a_i\left(\frac{c_i(0)}{1 - 1/\eta_i(\rho_i)}\right) < \sum_i a_i(0+).$$

Combining the above two inequalities, we have $D(\delta) = \sum_i D_i(\delta) < \bar{\delta}/(\delta + \kappa)$. Thus,

$$\lim_{\delta \uparrow \sum_i a_i(0+)} D(\delta) \leq \frac{\bar{\delta}}{\sum_i a_i(0+) + \kappa} < \frac{\sum_i a_i(0+)}{\sum_i a_i(0+) + \kappa} = \lim_{\delta \uparrow \sum_i a_i(0+)} \frac{\delta}{\delta + \kappa},$$

regardless of whether $\sum_i a_i(0+)$ is finite or infinite.[5] We complete the proof of claim (ii). ∎

Theorem 1 *There exists a unique positive pure strategy Nash equilibrium price vector $\mathbf{p}^* \in \mathcal{P}$. Furthermore, $\mathbf{p}^*$ satisfies $p_i^* > c_i(0)$ for all $i = 1, \cdots, n$.*

Proof: The result follows immediately from Propositions 5 and 6. Proposition 2 implies $p_i^* > c_i(0)$. ∎

---

[5]If $\sum_i a_i(0+) = \infty$, then the strict inequality still holds since $\bar{\delta}$ is always finite.

5. Convergence of Tatônnement Scheme

In this section, we show that the unique equilibrium is globally stable under the tatônnement scheme. Suppose each firm i chooses a best-response pricing strategy: choose p_i maximizing his profit $\pi_i(p_1, \ldots, p_n)$ while p_j's are fixed for all $j \neq i$. This section shows that the sequence of prices obtained by iterative application of this best-response strategy globally converges to the unique Nash equilibrium price vector (Theorem 2). The tatônnement convergence result not only shows the stability of the equilibrium, but also provides a computational method of finding it. We show the convergence rate is linear (Theorem 3), guaranteeing a certain degree of stability and computational efficiency.

In the tatônnement scheme we propose, a firm does not need to know the attraction functions of other firms. For the best response pricing strategy, he only needs to observe the aggregate attraction quantity of the other firms in each iteration, which can easily be deduced from its own demand and attraction value as well as κ.

In this section, we introduce additional assumptions on $C_i(\cdot)$ and $a_i(\cdot)$. We restrict them to be *continuously twice differentiable* to ensure the application of the implicit function theorem. This guarantees the existence of the derivatives of best-response functions. These additional assumptions are satisfied by most common cost and attraction functions.

By Theorem 1, there exists a unique equilibrium vector, which is denoted by $\mathbf{p}^* = (p_1^*, \cdots, p_n^*) \in \mathcal{P}$. Define $\mathcal{Q} := (0, a_1(0+)) \times \cdots \times (0, a_n(0+))$. Let $\mathbf{q}^* = (q_1^*, \cdots, q_n^*) \in \mathcal{Q}$ be the corresponding attraction vector where $q_i^* := a_i(p_i^*)$. Let $\widehat{q}_i := \sum_{j \neq i} q_j$ be the sum of attraction quantities of firms other than i. Set $\theta_i^* := q_i^*/(\widehat{q}_i^* + \kappa)$ and $d_i^* := q_i^*/(q_i^* + \widehat{q}_i^* + \kappa)$, which are both positive. Suppose we fix the price p_j for all $j \neq i$, and let $q_i := a_i(p_i)$ be the corresponding attraction.

Since a_i is one-to-one and $\delta = q_i + \widehat{q}_i$, condition (6) is equivalent to

$$\frac{c_i\left(\frac{q_i}{q_i + \widehat{q}_i + \kappa}\right)}{a_i^{-1}(q_i)} = 1 - \frac{1}{\eta_i \circ a_i^{-1}(q_i)}\left(1 + \frac{q_i}{\widehat{q}_i + \kappa}\right). \tag{8}$$

Using an argument similar to Proposition 3 and ensuing discussion, it can be shown that there is a unique solution q_i to (8) for each $\widehat{q}_i$ given by any positive number less than $\sum_{j \neq i} a_i(0+)$. (See Appendix A.2.) We call this solution q_i the *best response function* $\psi_i(\widehat{q}_i)$ for firm i. The unique equilibrium satisfies $\psi_i(\widehat{q}_i^*) = q_i^*$ where $\widehat{q}_i^* = \sum_{j \neq i} q_j^*$. Furthermore, it is easy to show that $\psi_i(\cdot)$ is strictly increasing. (See Appendix A.2 for the sketch of proof.)

Proposition 7 $\psi_i(\cdot)$ *is a strictly increasing function.*

¿From the definition of θ_i^* and $\psi_i(\widehat{q}_i^*) = q_i^*$, we know $\psi_i(\widehat{q}_i)/(\widehat{q}_i + \kappa) = \theta_i^*$ at $\widehat{q}_i = \widehat{q}_i^*$. The following proposition characterizes the relationship between $\psi_i(\widehat{q}_i)/(\widehat{q}_i + \kappa)$ and θ_i^*.

Proposition 8 $\psi_i(\widehat{q}_i)/(\widehat{q}_i + \kappa)$ *is strictly decreasing in* $\widehat{q}_i$*, and satisfies* $\psi_i(\widehat{q}_i^*)/(\widehat{q}_i^* + \kappa) = \theta_i^*$*. Thus,*

$$\frac{\psi_i(\widehat{q}_i)}{\widehat{q}_i + \kappa} \quad \begin{cases} > & \theta_i^*, \quad for \ \widehat{q}_i < \widehat{q}_i^* \\ = & \theta_i^*, \quad for \ \widehat{q}_i = \widehat{q}_i^* \\ < & \theta_i^*, \quad for \ \widehat{q}_i > \widehat{q}_i^*. \end{cases}$$

Furthermore, $\psi_i'(\widehat{q}_i)$ *is continuous, and satisfies* $0 < \psi_i'(\widehat{q}_i^*) < \theta_i^*$*.*

Proof: From the definition $\psi_i(\widehat{q}_i)$ and (8), we get

$$\frac{c_i\left(\frac{\psi_i(\widehat{q}_i)}{\psi_i(\widehat{q}_i) + \widehat{q}_i + \kappa}\right)}{a_i^{-1}(\psi_i(\widehat{q}_i))} = 1 - \frac{1}{\eta_i \circ a_i^{-1} \circ \psi_i(\widehat{q}_i)}\left(1 + \frac{\psi_i(\widehat{q}_i)}{\widehat{q}_i + \kappa}\right). \tag{9}$$

To see the existence of $\psi_i'(\cdot)$, we apply the *implicit function theorem* to

$$F(x, y) = \frac{c_i\left(\frac{x}{x+y+\kappa}\right)}{a_i^{-1}(x)} - 1 + \frac{1}{\eta_i \circ a_i^{-1}(x)}\left(1 + \frac{x}{y + \kappa}\right) = 0$$

with open domain, $(x, y) \in (0, a_i(0+)) \times (0, \sum_{j \neq i} a_i(0+))$. The Jacobian with respect to x is

$$F_x(x, y) = -(K_1(x, y) + J_1(x, y) + K_2(x, y) + J_2(x, y))$$

where

$$\begin{aligned} K_1(x, y) &:= -\frac{c_i'\left(\frac{x}{x+y+\kappa}\right)(y + \kappa)}{a_i^{-1}(x) \cdot (x + y + \kappa)^2}, \\ K_2(x, y) &:= -\frac{1}{\eta_i \circ a_i^{-1}(x) \cdot (y + \kappa)}, \\ J_1(x, y) &:= \frac{c_i\left(\frac{x}{x+y+\kappa}\right)}{(a_i^{-1}(x))^2} \cdot \left(a_i^{-1}\right)'(x), \\ J_2(x, y) &:= \left(1 + \frac{x}{y + \kappa}\right) \cdot \frac{\left(\eta_i \circ a_i^{-1}\right)'(x)}{\left(\eta_i \circ a_i^{-1}(x)\right)^2}. \end{aligned}$$

By Conditions A and (B1), c_i is positive and increasing, a_i^{-1} is positive and strictly decreasing, and $\eta_i \circ a_i^{-1}$ is positive and decreasing. Thus, both $K_1(x, y) + K_2(x, y)$ and $J_1(x, y) + J_2(x, y)$

are strictly negative. Hence $F_x(x, y) > 0$ and the implicit function theorem confirms the existence of the continuous derivative, $\psi'(\cdot)$, locally. Differentiating equation (9) with respect to $\widehat{q}_i$ yields

$$-K_1(\psi_i(\widehat{q}_i), \widehat{q}_i) \cdot \left(\psi_i'(\widehat{q}_i) - \frac{\psi_i(\widehat{q}_i)}{\widehat{q}_i + \kappa} \right) - J_1(\psi_i(\widehat{q}_i), \widehat{q}_i) \cdot \psi_i'(\widehat{q}_i)$$
$$= K_2(\psi_i(\widehat{q}_i), \widehat{q}_i) \cdot \left(\psi_i'(\widehat{q}_i) - \frac{\psi_i(\widehat{q}_i)}{\widehat{q}_i + \kappa} \right) + J_2(\psi_i(\widehat{q}_i), \widehat{q}_i) \cdot \psi_i'(\widehat{q}_i) .$$

Thus,

$$
\begin{aligned}
\psi_i'(\widehat{q}_i) &= \frac{K_1(\psi_i(\widehat{q}_i), \widehat{q}_i) + K_2(\psi_i(\widehat{q}_i), \widehat{q}_i)}{J_1(\psi_i(\widehat{q}_i), \widehat{q}_i) + J_2(\psi_i(\widehat{q}_i), \widehat{q}_i) + K_1(\psi_i(\widehat{q}_i), \widehat{q}_i) + K_2(\psi_i(\widehat{q}_i), \widehat{q}_i)} \cdot \frac{\psi_i(\widehat{q}_i)}{\widehat{q}_i + \kappa} \\
&< \frac{\psi_i(\widehat{q}_i)}{\widehat{q}_i + \kappa} ,
\end{aligned}
\tag{10}
$$

and it follows

$$\frac{d}{d\widehat{q}_i} \left(\frac{\psi_i(\widehat{q}_i)}{\widehat{q}_i + \kappa} \right) = \frac{\psi_i'(\widehat{q}_i)(\widehat{q}_i + \kappa) - \psi_i(\widehat{q}_i)}{(\widehat{q}_i + \kappa)^2} < 0. \tag{11}$$

Furthermore, (10) implies $\psi_i'(\widehat{q}_i)$ is continuous, and $\theta_i^* > \psi_i'(\widehat{q}_i^*)$ follows from the definition of θ_i^* and $\psi_i(\widehat{q}_i^*) = q_i^*$. Proposition 7 implies $\theta_i^* > 0$. ∎

Let $\mathbf{q} = (q_1, \ldots, q_n)$. We denote the vector of best response functions by $\Psi(\mathbf{q}) = (\psi_1(\widehat{q}_1), \cdots, \psi_n(\widehat{q}_n)) \in \mathcal{Q}$, where $\widehat{q}_i = \sum_{j \neq i} q_j$. Note that $\mathbf{q}^* = (q_1^*, \cdots, q_n^*)$ is a fixed point of Ψ, i.e., $\Psi(\mathbf{q}^*) = \mathbf{q}^*$. By Proposition 7, we have $\Psi(\mathbf{q}^1) < \Psi(\mathbf{q}^2)$ whenever two vectors $\mathbf{q}^1$ and $\mathbf{q}^2$ satisfy $\mathbf{q}^1 < \mathbf{q}^2$. (The inequalities are component-wise.) We now show that best-response pricing converges to the unique equilibrium. We define the sequence $\{\mathbf{q}^{(0)}, \mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \cdots\} \subset \mathcal{Q}$ by $\mathbf{q}^{(k+1)} := \Psi(\mathbf{q}^{(k)})$ for $k \geq 0$.

Theorem 2 *If each firm employs the best response strategy based on the prices of other firms in the previous iteration, the sequence of price vectors converges to the unique equilibrium price vector.*

Proof: Let $\mathbf{q}^{(0)} \in \mathcal{Q}$ denote the attraction vector associated with the initial price vector. Choose $\underline{\mathbf{q}}^{(0)}, \overline{\mathbf{q}}^{(0)} \in \mathcal{Q}$ such that $\underline{\mathbf{q}}^{(0)} < \mathbf{q}^{(0)} < \overline{\mathbf{q}}^{(0)}$ and $\underline{\mathbf{q}}^{(0)} < \mathbf{q}^* < \overline{\mathbf{q}}^{(0)}$. Such $\underline{\mathbf{q}}^{(0)}$ and $\overline{\mathbf{q}}^{(0)}$ exist since $\mathcal{Q}$ is a box-shaped open set.

For each $k \geq 0$, we define $\underline{\mathbf{q}}^{(k+1)} := \Psi(\underline{\mathbf{q}}^{(k)})$ and $\overline{\mathbf{q}}^{(k+1)} := \Psi(\overline{\mathbf{q}}^{(k)})$. From the monotonicity of $\Psi(\cdot)$ (Proposition 7) and $\Psi(\mathbf{q}^*) = \mathbf{q}^*$, we get

$$\underline{\mathbf{q}}^{(k)} < \mathbf{q}^{(k)} < \overline{\mathbf{q}}^{(k)} \quad \text{and} \quad \underline{\mathbf{q}}^{(k)} < \mathbf{q}^* < \overline{\mathbf{q}}^{(k)}. \tag{12}$$

Let $u^{(k)} := \max_i \left\{ \widehat{\overline{q}}_i^{(k)} \big/ \widehat{q}_i^* \right\}$. Clearly, $u^{(k)} > 1$ for all k by (12). We show that the sequence $\left\{ u^{(k)} \right\}_{k=0}^{\infty}$ is strictly decreasing. For each i,

$$\overline{q}_i^{(k+1)} = \psi_i(\widehat{\overline{q}}_i^{(k)}) < \left(\widehat{\overline{q}}_i^{(k)} + \kappa \right) \cdot \theta_i^* = \left(\widehat{\overline{q}}_i^{(k)} + \kappa \right) \cdot \frac{q_i^*}{\widehat{q}_i^* + \kappa} \leq \frac{\widehat{\overline{q}}_i^{(k)}}{\widehat{q}_i^*} \cdot q_i^* \leq u^{(k)} q_i^* \,,$$

where the first inequality comes from Proposition 8, the second one from (12), and the last one from the definition of $u^{(k)}$. Thus

$$\widehat{\overline{q}}_i^{(k+1)} = \sum_{j \neq i} \overline{q}_j^{(k+1)} < \sum_{j \neq i} u^{(k)} q_j^* = u^{(k)} \widehat{q}_i^* \,, \tag{13}$$

and $u^{(k+1)} = \max_i \{ \widehat{\overline{q}}_i^{(k+1)} / \widehat{q}_i^* \} < u^{(k)}$.

Since $\left\{ u^{(k)} \right\}_{k=0}^{\infty}$ is a monotone and bounded sequence, it converges. Let $u^{\infty} := \lim_{k \to \infty} u^{(k)}$. We claim $u^{\infty} = 1$. Suppose, by way of contradiction, $u^{\infty} > 1$. By Proposition 8, $\psi_i(\widehat{q}_i)/(\widehat{q}_i + \kappa)$ is strictly decreasing in $\widehat{q}_i$. Thus, for any $\widehat{q}_i \geq \frac{1}{2}(1 + u^{\infty}) \cdot \widehat{q}_i^*$, there exists $\epsilon \in (0, 1)$ such that for each i, we have

$$\frac{\psi_i(\widehat{q}_i)}{(\widehat{q}_i + \kappa)} \leq (1 - \epsilon) \cdot \frac{\psi_i(\widehat{q}_i^*)}{\widehat{q}_i^* + \kappa} = (1 - \epsilon) \cdot \frac{q_i^*}{\widehat{q}_i^* + \kappa} \,.$$

For any k, if $\widehat{\overline{q}}_i^{(k)} \geq \frac{1}{2}(1 + u^{\infty}) \cdot \widehat{q}_i^*$, then

$$\overline{q}_i^{(k+1)} = \psi_i \left(\widehat{\overline{q}}_i^{(k)} \right) \leq \left(\widehat{\overline{q}}_i^{(k)} + \kappa \right) \cdot (1 - \epsilon) \cdot \frac{q_i^*}{\widehat{q}_i^* + \kappa}$$

$$\leq \frac{\widehat{\overline{q}}_i^{(k)}}{\widehat{q}_i^*} \cdot q_i^* \cdot (1 - \epsilon) \leq (1 - \epsilon) \cdot u^{(k)} \cdot q_i^* \,.$$

Otherwise, we have $\widehat{q}_i^* < \widehat{\overline{q}}_i^{(k)} < \frac{1}{2}(1 + u^{\infty}) \cdot \widehat{q}_i^*$. By Proposition 8,

$$\overline{q}_i^{(k+1)} = \psi_i \left(\widehat{\overline{q}}_i^{(k)} \right) < \left(\widehat{\overline{q}}_i^{(k)} + \kappa \right) \cdot \frac{q_i^*}{\widehat{q}_i^* + \kappa}$$

$$\leq \frac{\widehat{\overline{q}}_i^{(k)}}{\widehat{q}_i^*} \cdot q_i^* \leq \frac{1}{2}(1 + u^{\infty}) \cdot q_i^* \,.$$

Therefore, we conclude, using an argument similar to (13),

$$u^{(k+1)} \leq \max \left\{ (1 - \epsilon) \cdot u^{(k)}, \ (1 + u^{\infty})/2 \right\} \,.$$

¿From $u^{(k+1)} > u^{\infty} > (1 + u^{\infty})/2$, we obtain $u^{(k+1)} \leq (1 - \epsilon) \cdot u^{(k)}$, implying $u^{(k)} \to 1$ as $k \to \infty$. This is a contradiction.

Similarly, we can show that $l^{(k)} := \min_i \left\{ \widehat{\underline{q}}_i^{(k)} \, / \, \widehat{q}_i^* \right\}$ is a strictly increasing sequence converging to 1. ∎

The following Corollary identifies a sufficient condition for the monotonicity of the sequence of attraction vectors (and also for price vectors).

Corollary 1 *If* $\mathbf{q}^{(0)} > \Psi(\mathbf{q}^{(0)})$ *or* $\mathbf{q}^{(0)} < \Psi(\mathbf{q}^{(0)})$, *then* $\{\mathbf{q}^{(k)}\}_{k \geq 0}$ *is monotone.*

Proof: Suppose $\mathbf{q}^{(0)} > \Psi(\mathbf{q}^{(0)}) = \mathbf{q}^{(1)}$. We proceed by induction. Assume that $\mathbf{q}^{(j)} > \mathbf{q}^{(j+1)}$ for all $j < k$. Then, $\mathbf{q}^{(k)} - \mathbf{q}^{(k+1)} = \Psi(\mathbf{q}^{(k-1)}) - \Psi(\mathbf{q}^{(k)}) > \mathbf{0}$ since $\Psi(\cdot)$ is strictly increasing and $\mathbf{q}^{(k-1)} > \mathbf{q}^{(k)}$. ∎

The following proposition shows the linear convergence of tatônnement in the space of attraction values.

Proposition 9 *The sequence* $\{\mathbf{q}^{(k)}\}_{k \geq 0}$ *converges linearly.*

Proof: Consider $\left\{\underline{\mathbf{q}}^{(k)}\right\}_{k=0}^{\infty}$ and $\left\{\overline{\mathbf{q}}^{(k)}\right\}_{k=0}^{\infty}$ in the proof of Theorem 2. Recall $\underline{\mathbf{q}}^{(k)} < \mathbf{q}^{(k)} < \overline{\mathbf{q}}^{(k)}$ and $\underline{\mathbf{q}}^{(k)} < \mathbf{q}^* < \overline{\mathbf{q}}^{(k)}$. We will show that $\underline{\mathbf{q}}^{(k)}$ and $\overline{\mathbf{q}}^{(k)}$ converges to $\mathbf{q}^*$ linearly. Since $\mathcal{Q}$ is a box-shaped open set, there exists a convex compact set $\mathcal{B} \subset \mathcal{Q}$ containing all elements of $\left\{\underline{\mathbf{q}}^{(k)}\right\}_{k=0}^{\infty}$ and $\left\{\overline{\mathbf{q}}^{(k)}\right\}_{k=0}^{\infty}$. From (11) in the proof of Proposition 8, there exists $\delta > 0$ such that for any $\mathbf{q} \in \mathcal{B}$, we have

$$\frac{d}{d\widehat{q}_i} \left(\frac{\psi(\widehat{q}_i)}{\widehat{q}_i + \kappa} \right) \leq -\delta.$$

¿From integrating both sides of the above expression from $\widehat{q}_i^*$ to $\widehat{\overline{q}}_i^{(k)}$,

$$\frac{\psi_i(\widehat{\overline{q}}_i^{(k)})}{\widehat{\overline{q}}_i^{(k)} + \kappa} - \frac{\psi(\widehat{q}_i^*)}{\widehat{q}_i^* + \kappa} \leq -\delta \left(\widehat{\overline{q}}_i^{(k)} - \widehat{q}_i^* \right)$$

since the line segment connecting $\mathbf{q}^*$ and $\overline{\mathbf{q}}^{(k)}$ lies within $\mathcal{B}$.

Define $\delta_1 := \delta \cdot \min_i \min_{\mathbf{q} \in \mathcal{B}} \left\{ (\widehat{q}_i + \kappa) \cdot \widehat{q}_i^* / q_i^* \right\} > 0$. We choose $\delta > 0$ is sufficiently small such that $\delta_1 < 1$. Recall $\overline{q}_i^{(k+1)} = \psi_i(\widehat{\overline{q}}_i^{(k)})$ and $q_i^* = \psi(\widehat{q}_i^*)$. Rearranging the above inequality and multiplying it by $(\widehat{\overline{q}}_i^{(k)} + \kappa)/q_i^*$,

$$\begin{aligned}
\overline{q}_i^{(k+1)}/q_i^* &\leq (\widehat{\overline{q}}_i^{(k)} + \kappa)/(\widehat{q}_i^* + \kappa) - \delta \cdot (\widehat{\overline{q}}_i^{(k)} - \widehat{q}_i^*) \cdot (\widehat{\overline{q}}_i^{(k)} + \kappa)/q_i^* \\
&\leq \widehat{\overline{q}}_i^{(k)}/\widehat{q}_i^* - \delta_1 \cdot (\widehat{\overline{q}}_i^{(k)}/\widehat{q}_i^* - 1) \\
&= (1 - \delta_1) \cdot \widehat{\overline{q}}_i^{(k)}/\widehat{q}_i^* + \delta_1.
\end{aligned}$$

where the second inequality comes from $\widehat{\overline{q}}_i^{(k)} > \widehat{q}_i^*$ and the definition of δ_1.

Let $\rho(k) := \max_i \{\overline{q}_i^{(k)}/q_i^*\}$. Thus, $\overline{q}_j^{(k)} \le \rho(k) \cdot q_j^*$ holds for all j, and summing this inequality for all $j \ne i$, we get $\widehat{\overline{q}}_i^{(k)} \le \rho(k) \cdot \widehat{q}_i^*$. Thus, $\overline{q}_i^{(k+1)}/q_i^*$ is bounded above by $(1 - \delta_1) \cdot \rho(k) + \delta_1$ for each i, and we obtain

$$\rho(k+1) \le (1 - \delta_1) \cdot \rho(k) + \delta_1 .$$

Using induction, it is easy to show

$$\rho(k) \le (1 - \delta_1)^k \cdot (\rho(0) - 1) + 1$$

Therefore, we obtain

$$\max_i \left\{ (\overline{q}_i^{(k)} - q_i^*)/q_i^* \right\} = \rho(k) - 1 \le (1 - \delta_1)^k \cdot (\rho(0) - 1)$$
$$= (1 - \delta_1)^k \cdot \max_i \left\{ (\overline{q}_i^{(0)} - q_i^*)/q_i^* \right\} \qquad \text{and}$$
$$\max_i \left\{ \overline{q}_i^{(k)} - q_i^* \right\} \le (1 - \delta_1)^k \cdot \max_i \{q_i^*\} \cdot \max_i \left\{ (\overline{q}_i^{(0)} - q_i^*)/q_i^* \right\} ,$$

showing the linear convergence of the upper bound sequence $\overline{\mathbf{q}}^{(k)}$. (See Bertsekas (1995).) A similar argument shows the linear convergence of $\underline{\mathbf{q}}^{(k)}$. ∎

The linear convergence in the above proposition is not with respect to the price vector but with respect to the attraction vector, i.e. not in $\mathbf{p}$ but in $\mathbf{q}$. Yet, the following theorem also shows the linear convergence with respect to the price vector. Let $\{\mathbf{p}^{(k)}\}_{k \ge 0}$ be the sequence defined by $p_i^{(k)} := a_i^{-1}(q_i^{(k)})$.

Theorem 3 *The sequence $\{\mathbf{p}^{(k)}\}_{k \ge 0}$ converges linearly.*

Proof: Consider $\{\underline{\mathbf{q}}^{(k)}\}_{k=0}^{\infty}$ and $\{\overline{\mathbf{q}}^{(k)}\}_{k=0}^{\infty}$ in the proof of Proposition 9. Let $\{\underline{\mathbf{p}}^{(k)}\}_{k=0}^{\infty}$ and $\{\overline{\mathbf{p}}^{(k)}\}_{k=0}^{\infty}$ be the corresponding sequences of price vectors. Within the compact interval $[\overline{p}_i^{(0)}, \underline{p}_i^{(0)}]$, the derivative of a_i is continuous and its infimum is strictly negative. By the Inverse Function Theorem, the derivative of $a_i^{-1}(\cdot)$ is continuous in the compact domain of $[\underline{q}_i^{(0)}, \overline{q}_i^{(0)}]$. Recall $p_i^* = a_i^{-1}(q_i^*)$. There exists some bound $M > 0$ such that

$$|p_i - p_i^*| = |a_i^{-1}(q_i) - a_i^{-1}(q_i^*)| \le M \cdot |q_i - q_i^*|$$

whenever $a_i(p_i) = q_i \in [\underline{q}_i^{(0)}, \overline{q}_i^{(0)}]$ for all i. From the proof of Proposition 9, we get $q_i^{(k)} \in (\underline{q}_i^{(k)}, \overline{q}_i^{(k)}) \subset [\underline{q}_i^{(0)}, \overline{q}_i^{(0)}]$. Therefore, the linear convergence of $\{\mathbf{q}^{(k)}\}_{k \ge 0}$ implies the linear convergence of $\{\mathbf{p}^{(k)}\}_{k \ge 0}$. ∎

6. Numerical Results

In this section, we perform the computation of the tatônnement scheme using Example 1, and report on its convergence behavior.

We first consider the price competition among $n = 5$ firms. We use generalized logit attraction functions. Thus,

$$d_i(\mathbf{p}) = \frac{\exp(-\lambda_i p_i)}{\sum_{j=1}^n \exp(-\lambda_j p_j) + \kappa}$$

where the parameter $\lambda_i > 0$ depends on each firm i. We used $\kappa = 0$, 0.01, 0.05, 0.1, 0.5 values. For stochastic demand, we use the additive demand uncertainty model $\delta(d_i, \varepsilon) = d_i + \varepsilon_i$, where ε_i has a Gaussian distribution with mean 0 and variance 0.2. The cost function in Example 1 is given by

$$C_i'(d_i) = w_i - h_i P(y_i \geq d_i + \varepsilon_i) + b_i P(y_i \leq d_i + \varepsilon_i), \tag{14}$$

where $w_i = 1.0$, $h_i = 0.2$, and $b_i = 2.0$ for each firm i. The value of y_i depends on firm i. We initialize the algorithm with $\mathbf{p}^{(0)} = (C_1'(0), \ldots, C_n'(0))$. The algorithm is terminated when the Euclidean distance between two successive $\mathbf{p}^{(k)}$ vectors is less than tolerance 10^{-8}. We use Matlab 5.3 on a PC notebook with 900 megahertz processor speed and 384 megabytes of memory. A typical iteration takes about several seconds, most of which is spent invoking the Gaussian cumulative density function in (14).

Table 2 shows the equilibrium price vector $\mathbf{p}^*$ and expected demand vector $\mathbf{d}^*$ provided $(\lambda_1, \ldots, \lambda_5) = (0.5, 0.75, 1, 1.25, 1.5)$ and $(y_1, \ldots, y_5) = (0.5, 0.6, 0.7, 0.8, 0.9)$. The column marked by $\sum_i d_i^*$ is the total expected demand as a function of κ. The last column shows the number of iterations until termination. We repeat computation with $(\lambda_1, \ldots, \lambda_5) = (1.5, 1.25, 1, 0.75, 0.5)$, and report results in Table 3.

Using the same λ vector and y vector as in Table 2, Figure 1 plots $\|\mathbf{q}^{(k)} - \mathbf{q}^*\|$ as a function of iteration count, for each value of κ. It shows linear convergence as predicted by Proposition 9. The convergence rate of the price vector is also linear as illustrated by Figure 1. This figure indicates that the number of required iterations is smaller when there are more competing firms.

We now vary the number of firms in competition, and use $n = 2$, 5, 10. For each n, the set of λ_i values and y_i values is given in Table 4. We fix κ at 0.01. We obtain similar linear convergence results, shown in Figure 2.

Table 2: Tatônnement Computation Output. Generalized logit attraction model. $(\lambda_1, \ldots, \lambda_5) = (0.5, 0.75, 1, 1.25, 1.5)$. $(y_1, \ldots, y_5) = (0.5, 0.6, 0.7, 0.8, 0.9)$.

κ	$\mathbf{p}^*$	$\mathbf{d}^*$	$\sum_i d_i^*$	Iter
0	(3.6987, 2.5936, 2.0659, 1.7702, 1.5826)	(0.2499, 0.2271, 0.2013, 0.1738, 0.1479)	1	14
0.01	(3.6807, 2.5837, 2.0602, 1.7666, 1.5801)	(0.2467, 0.2238, 0.1980, 0.1708, 0.1452)	0.9844	14
0.05	(3.6170, 2.5490, 2.0401, 1.7538, 1.5714)	(0.2348, 0.2118, 0.1862, 0.1600, 0.1356)	0.9283	13
0.1	(3.5512, 2.5137, 2.0198, 1.7409, 1.5626)	(0.2219, 0.1989, 0.1738, 0.1487, 0.1257)	0.8689	12
0.5	(3.2722, 2.3687, 1.9366, 1.6879, 1.5265)	(0.1582, 0.1375, 0.1172, 0.0985, 0.0823)	0.5937	9

Table 3: Tatônnement Computation Output. Generalized logit attraction model. $(\lambda_1, \ldots, \lambda_5) = (1.5, 1.25, 1, 0.75, 0.5)$. $(y_1, \ldots, y_5) = (0.5, 0.6, 0.7, 0.8, 0.9)$.

κ	$\mathbf{p}^*$	$\mathbf{d}^*$	$\sum_i d_i^*$	Iter
0	(1.6427, 1.7932, 2.0625, 2.5429, 3.5335)	(0.1334, 0.1666, 0.1993, 0.2328, 0.2679)	1	14
0.01	(1.6391, 1.7890, 2.0569, 2.5347, 3.5192)	(0.1313, 0.1639, 0.1961, 0.2292, 0.2641)	0.9846	14
0.05	(1.6264, 1.7743, 2.0376, 2.5055, 3.4686)	(0.1236, 0.1542, 0.1847, 0.2164, 0.2502)	0.9291	13
0.1	(1.6136, 1.7594, 2.0179, 2.4757, 3.4164)	(0.1154, 0.1440, 0.1726, 0.2028, 0.2353)	0.8701	12
0.5	(1.5613, 1.6989, 1.9362, 2.3498, 3.1930)	(0.0779, 0.0969, 0.1169, 0.1391, 0.1642)	0.5948	9

Table 4: Tatônnement Computation. Generalized logit attraction model. Input Data for Figure 2

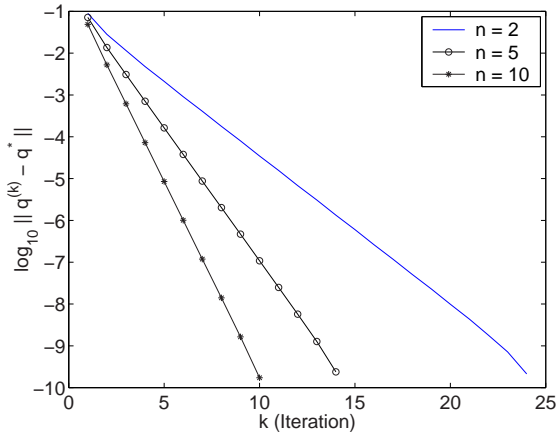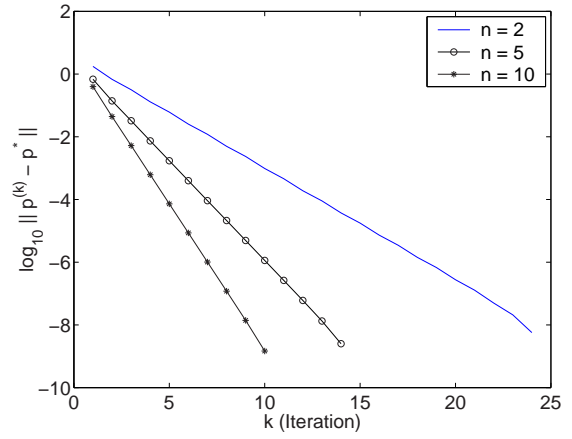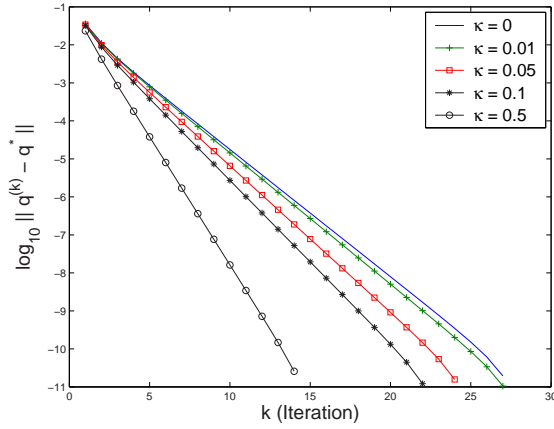| n	$(\lambda_1, \ldots, \lambda_n)$	$(y_1, \ldots, y_n)$
2	(0.5, 1.5)	(0.5, 0.9)
5	(0.5, 0.75, 1, 1.25, 1.5)	(0.5, 0.6, 0.7, 0.8, 0.9)
10	(0.5, 0.6, 0.7, 0.8, 0.9, 1.1, 1.2, 1.3, 1.4, 1.5)	(0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9)
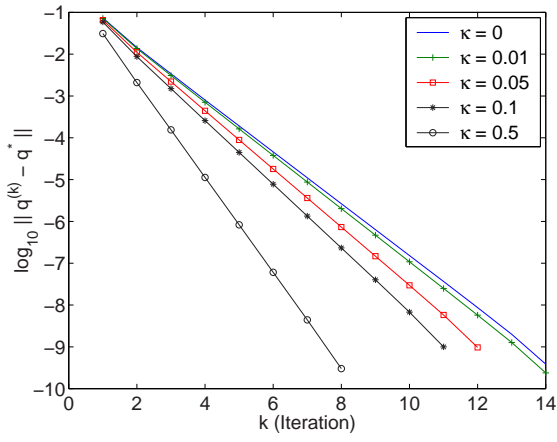
(a) Convergence of $\mathbf{q}^{(k)}$.
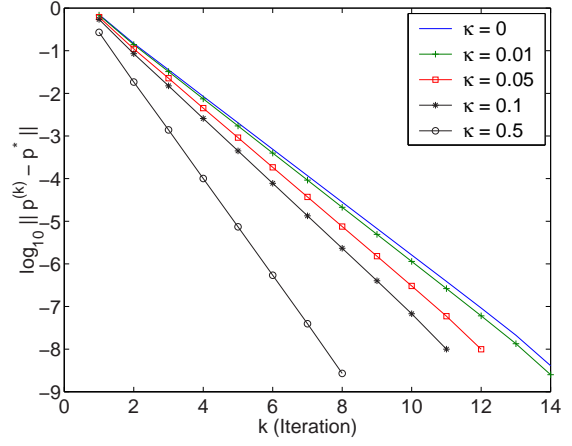
(b) Convergence of $\mathbf{p}^{(k)}$.

Figure 1: Generalized logit attraction model. Initialization $\mathbf{p}^{(0)} = (C_1'(0), \ldots, C_n'(0))$. $n = 5$.
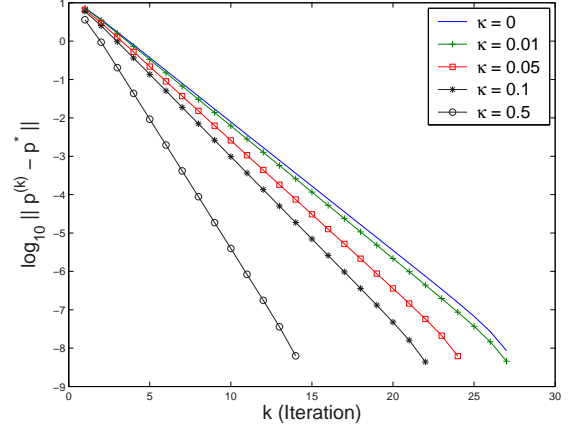

(a) Convergence of $\mathbf{q}^{(k)}$.

(b) Convergence of $\mathbf{p}^{(k)}$.

Figure 2: Generalized logit attraction model. Initialization $\mathbf{p}^{(0)} = (C_1'(0), \ldots, C_n'(0))$. $\kappa = 0.01$.
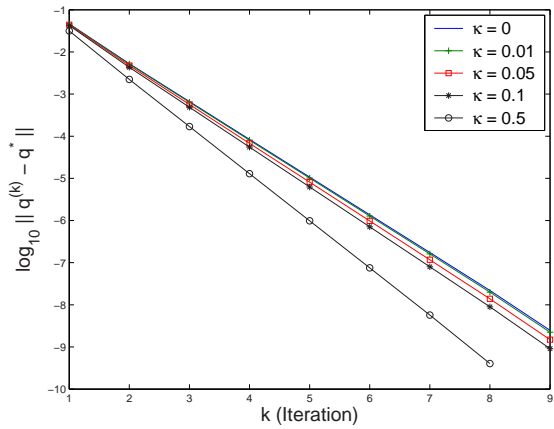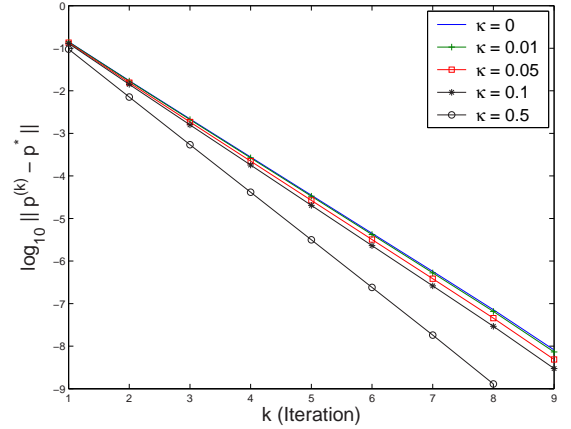
(a) Convergence of $\mathbf{q}^{(k)}$.

(b) Convergence of $\mathbf{p}^{(k)}$.

Figure 3: Cobb-Douglas attraction model, i.e., $d_i(\mathbf{p}) = (p_i^{-\gamma_i})/(\sum_{j=1}^{n} p_j^{-\gamma_j} + \kappa)$, where $\boldsymbol{\gamma} = (1.1, 1.6, 2.1, 2.6, 3.1)$. Initialization $\mathbf{p}^{(0)} = (C_1'(0), \ldots, C_n'(0))$. $n = 5$.

(a) Convergence of $\mathbf{q}^{(k)}$.

(b) Convergence of $\mathbf{p}^{(k)}$.

Figure 4: Linear attraction model, i.e., $d_i(\mathbf{p}) = (\alpha_i - \beta_i p_i)/(\sum_{j=1}^{n}(\alpha_j - \beta_j p_j) + \kappa)$ where $\boldsymbol{\beta} = (0.1, 0.3, 0.5, 0.7, 0.9)$ and the ratios $\alpha_i/(\beta_i \cdot c_i(1))$'s are given by $(1.01, 1.02, 1.03, 1.04, 1.05)$. Initialization $\mathbf{p}^{(0)} = (C_1'(0), \ldots, C_n'(0))$. $n = 5$.

Additional tests have been carried out for the Cobb-Douglas and Linear attraction functions. See Figures 3 and 4.

A. Appendix

A.1 Continuity of $D_i(\delta)$

By the continuity assumptions on all involved functions, $f_i(x_i, \delta) := L_i(x_i|\delta) - R_i(x_i|\delta)$ is continuous in (x_i, δ). For each $\delta \in \Delta$, there is a unique $D_i(\delta)$ satisfying $f_i(D_i(\delta), \delta) = 0$ by Proposition 3. Furthermore, the argument following Proposition 3 shows $D_i(\delta)$ is strictly increasing in δ.

Consider any sequence $\delta_k \to \delta_0$. If $\lim_{k \to \infty} D_i(\delta_k)$ exists, then by the continuity of f, we have

$$ 0 = \lim_{k \to \infty} f(D_i(\delta_k), \delta_k) = f\big(\lim_{k \to \infty} D_i(\delta_k), \lim_{k \to \infty} \delta_k \big) = f\big(\lim_{k \to \infty} D_i(\delta_k), \delta_0 \big), $$

implying $\lim_{k \to \infty} D_i(\delta_k) = D_i(\delta_0)$. Assuming $\lim_{k \to \infty} D_i(\delta_k)$ does not exist, apply the above argument to subsequences converging to $\limsup_{k \to \infty} D_i(\delta_k)$ and $\liminf_{k \to \infty} D_i(\delta_k)$, respectively. Then, their limits should be the same. Thus, $D_i(\delta)$ is continuous in δ.

A.2 Existence and Monotonicity of ψ_i

Since $a_i(\cdot)$ is one-to-one and $\delta = q_i + \widehat{q}_i$, condition (6) is equivalent to (8). Since $c_i(\cdot)$ is nondecreasing, and $a_i^{-1}(\cdot)$ is strictly increasing, the left side of (6) increases strictly. Since $\eta_i \circ a_i^{-1}(\cdot)$ is nonincreasing and $1 + \frac{\cdot}{\widehat{q}_i + \kappa}$ is strictly increasing, the right side of (6) decreases strictly. As q_i approaches 0 and $a_i(0)$, both limits are the same as those in the proof of Proposition 3. Thus $\psi_i(\cdot)$ is well-defined.

Now the lifting-and-dropping argument following Proposition 3 can be applied to prove Proposition 7, which shows the monotonicity of $\psi_i(\cdot)$.

References

Allon, G., and A. Federgruen. 2004. Service Competition with General Queueing Facilities. *Working Paper*.

Anderson, S. P., A. De Palma, and J.-F. Thisse. 1996. *Discrete choice theory of product differentiation.* The MIT Press.

Benassy, J.-P. 1989. Market Size and Substitutability in Imperfect Competition: A Bertrand-Edgeworth-Chamberlin Model. *Review of Economic Studies* 56:217–234.

Bernstein, F., F. Chen, and A. Federgruen. 2002. Vendor Managed Inventories and Supply Chain Coordination: The Case with One Supplier and Competing Retailers. *Working Paper*.

Bernstein, F., and A. Federgruen. 2003. Pricing and Replenishment Strategies in a Distribution System with Competing Retailers. *Operations Research* 51 (3): 409–426.

Bernstein, F., and A. Federgruen. 2004a. Comparative Statics, Strategic Complements and Substitutes in Oligopolies. *Journal of Mathematical Economics* 40 (6): 713–746.

Bernstein, F., and A. Federgruen. 2004b. A General Equilibrium Model for Industries with Price and Service Competition. *Operations Research* 52 (6): 868–886.

Bertrand, J. 1883. Théorie Mathématique de la Richesse Sociale. *Journal des Savants*:499–508.

Bertsekas, D. P. 1995. *Nonlinear programming*. Athena Scientific.

Besanko, D., S. Gupta, and D. Jain. 1998. Logit Demand Estimation Under Competitive Pricing Behavior: An Equilibrium Framework. *Management Science* 44:1533–1547.

Cachon, G. P., and P. T. Harker. 2002. Competition and Outsourcing with Scale Economies. *Management Science* 48 (10): 1314–1333.

Caplin, A., and B. Nalebuff. 1991. Aggregation and Imperfection Competition: On the Existence of Equilibrium. *Econometrica* 59:25–59.

Chamberlin, E. H. 1933. *The theory of monopolistic competition*. Harvard University Press.

Cournot, A. 1838. *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. Macmillan. English edition (ed. N. Bacon), 1863: *Researches into the Mathematical Principles of the Theory of Wealth*.

Debreu, G. 1960. Book Review of *Individual Choice Behavior: A Theoretical Analysis. The American Economic Review* 50:188–190.

Dubey, P., O. Haimanko, and A. Zapechelnyuk. 2003. Strategic Complements and Substitutes, and Potential Games. *Working Paper*.

Edgeworth, F. 1922. The mathematical economies of Professor Amoroso. *Economic Journal* 32:400–407.

Edgeworth, F. 1925. The Pure Theory of Monopoly. In *Papers Relating to Political Eonomy*, 111–142.

Friedman, J. 1983. *Oligopoly theory*. Cambridge University Press.

Hotelling, H. 1929. Stability in Competition. *Economic Journal* 39:41–57.

Luce, R. D. 1959. *Individual choice behavior: A theoretical analysis*. Wiley.

Mahajan, S., and G. Van Ryzin. 1998. Supply Chain Contracting and Coordination with Stochastic Demand. In *Quantitive Models for Supply Chain Management*, ed. S. Tayur, M. Magazine, and R. Ganeshan. Kluwer.

Medhi, J. 2003. *Stochastic models in queueing theory*. Second ed. Academic Press.

Milgrom, P., and J. Roberts. 1990. Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities. *Econometrica* 58:1255–1277.

Milgrom, P., and C. Shannon. 1994. Monotone Comparative Statics. *Econometrica* 62:157–180.

Mizuno, T. 2003. On the Existence of a Unique Price Equilibrium for Models of Product Differentiation. *International Journal of Industrial Organization* 62:761–793.

Roberts, J., and H. Sonnenschein. 1977. On the Foundations of the Theory of Monopolistic Competition. *Econometrica* 45:101–113.

Robinson, J. 1933. *The economics of imperfect competition*. Macmillan.

So, K. C. 2000. Price and Time Competition for Service Delivery. *Manufacturing & Service Operations Management* 2:392–407.

Topkis, D. M. 1979. Equilibrium Points in Nonzero-Sum n-Person Submodular Games. *SIAM Journal on Control and Optimization* 17:773–787.

Vives, X. 1985. On the Efficiency of Bertrand and Cournot Equilibria with Product Differentiation. *Journal of Economic Theory* 36:166–175.

Vives, X. 1990. Nash Equilibrium with Strategic Complementarities. *Journal of Mathematical Economics*:305–321.

Vives, X. 1999. *Oligopoly pricing: Old ideas and new tools.* The MIT Press.