Lecture notes on risk management, public policy, and the financial system

Market equilibrium and relative risk

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Portfolios and investment choices

- Available investment choices can be expanded by mixing assets in **portfolios**
- Simple approach to identifying available investment choices
  - What combinations of **portfolio expected return** and **portfolio return variance** or **volatility**—representing risk—are available?
  - Are any of these combinations clearly superior or inferior to others?
- Based on expected returns, volatilities and correlations of constituent assets
- **Example:** Facebook Inc. (ticker FB) and Coca-Cola Co. (KO) 18May2012 to 24Sep2020

	FB	KO
Mean daily logarithmic return (%)	0.090177	0.012446
Standard deviation of daily returns (%)	2.346570	1.141170
Correlation coefficient	0.21290	

- Once we understand menu of available and reasonable choices clearly, we can analyze which ones investors *prefer*

Portfolio expected return

- The **portfolio expected return** is a simple weighted average of the constituent assets' expected returns:
- In the case of just two constituents:

$$\mu_p = w\mu_1 + (1 - w)\mu_2,$$

with $w \equiv$ asset 1 weight

- Portfolio expected return changes proportionally to a change in constituent expected return
- **Example:** the expected return of a 50-50 FB-KO portfolio is

$$\mu_p = 0.5 \cdot 0.000902 + 0.5 \cdot 0.000124 = 0.00051311$$

or 5.13 basis points per day

Portfolio return variance and volatility

- The **portfolio return variance** is *not* a weighted average of the constituent variances:

$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2\rho_{12},$$

  - **Portfolio volatility** is its square root: $\sigma_p = \sqrt{\sigma_p^2}$
- Portfolio variance and volatility do *not* change proportionally to a change in constituent volatility
- And the portfolio variance can be strongly influenced up or down by return correlation
- **Example:** the return variance of a 50-50 FB-KO portfolio is

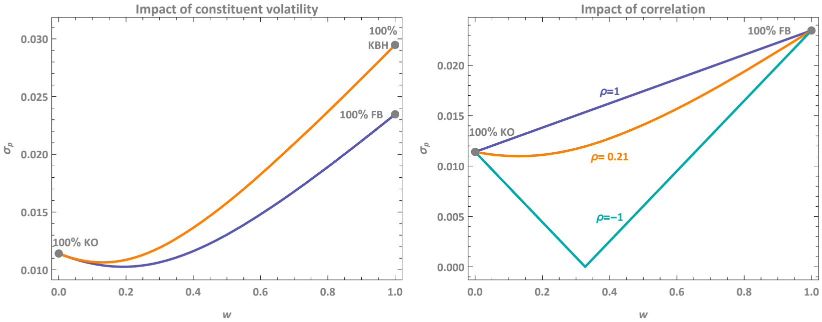
$$\sigma_p^2 = 0.5^2 \cdot 0.000551 + 0.5^2 \cdot 0.000130 + 0.5 \cdot 0.5 \cdot 0.0235 \cdot 0.0114 \cdot 0.2129 = 0.000199$$

and the portfolio volatility is $\sqrt{0.000199} = 0.01409690$ or 1.410 percent daily.

Diversification is powerful

- **Diversification:** combining assets can lead to a reduction of risk without sacrificing return
- Diversification expands investors' opportunity set
  - Adding a small amount of even a high-volatility asset can reduce portfolio volatility
  - But effect more limited if return correlation strongly positive
- Lower correlation enables investor to achieve lower portfolio volatility for any given expected return
  - Negative correlation provides the strongest volatility reduction
  - Mixing risky assets can reduce portfolio return volatility even if correlation is positive

Impact of diversification on portfolio return volatility



Left panel: volatility (y-axis) of portfolios combining long positions in KO stock with long positions in FB and KB Home (KBH), assuming a daily return correlation of 0, both plotted as a function of the FB or KBH portfolio weight (x-axis). Right panel: volatility of portfolios combining long positions in KO stock with long positions in FB assuming different non-zero return correlations.

Feasible and efficient portfolios

- Not every **feasible** or **attainable** portfolio is **efficient**
  - For each portfolio, find return and volatility
  - Two portfolios may have same volatility but different returns (or v.v.)
  - Portfolio with same volatility but lower return—or same return but higher volatility—than some other is not efficient
- **Efficient frontier:** return and volatility points of efficient portfolios
  - Traces risk-return tradeoff in mean-variance framework
- **Global minimum variance portfolio** has lowest return and volatility among efficient portfolios
- ($\rightarrow$)**Risk-free assets** may also be available for inclusion
  - Have non-zero return (usually but not always positive) but zero volatility

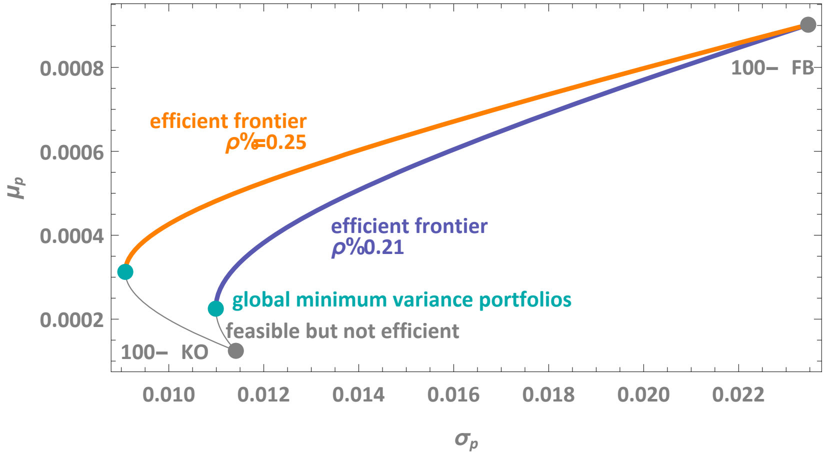
Feasible and efficient portfolios: example

- Portfolio consisting of all or mostly low-return/low-volatility KO not efficient
- Adding *some* high-return/high-volatility FB lowers portfolio volatility and raises portfolio return
  - Unambiguously more desirable to investors than KO alone
- Portfolios with high share of FB have higher volatility and return
  - May be more desirable to some investors

FB wt. (%)	KO wt. (%)	return (%)	volatility (%)
0	100	0.012446	1.141170
10	90	0.020219	1.101140
12.92	87.08	0.022486	1.098950
20	80	0.027992	1.111820
50	50	0.051311	1.409690
90	10	0.082404	2.139120
100	0	0.090177	2.346570

In percent. The global minimum variance portfolio is highlighted.

The risk-return tradeoff



Volatility (x-axis) and mean (y-axis) of portfolios combining long positions in KO and FB stock. Purple plot shows feasible portfolios estimated using the historical return correlation of 0.2129 . The heavy part of the plot is the efficient frontier. Orange plot shows efficient frontier if the return correlation were -0.25.

Explaining equilibrium asset prices and returns

- Market-clearing process determines asset prices and prospective returns by finding **equilibrium price**, given supply and demand schedules for securities
  - Assumptions about investors determine demand schedules
- Steps in the explanation:
  1. Take investment choices/prospective returns as given, analyze from individual point of view:
     - 1.1 Identify **efficient portfolios:** portfolios that don't waste opportunities
     - 1.2 Explain how individuals choose among efficient portfolios
  2. Once we know how individuals choose, how does market clear and establish the prospective returns individuals face?
- **Mean-variance framework:** payoffs on individual risky asset depend only on mean return and volatility

Investor preferences and risk

- Problems for quantitative definition of risk arise from preferences as well
- Mathematical optimization requires unambiguous preference ranking of sets of choices, portfolios
- Even with well-defined probability distribution of outcomes, difficulties in obtaining
  - Unambiguous preference ranking
  - Useful definition of **risk aversion**
- **Expected utility** axioms: require specification of **utility function**
- Approaches include

  Mean-variance dominance: provides limited ability to rank outcomes, doesn't consider tail returns

  Stochastic dominance looks at entire probability distribution

  - Approaches may contradict one another and may fail to provide unambiguous ranking

Choosing among portfolios

- Simple model: investor assumed to engage in **mean-variance optimization**
  - Happiness/wealth/utility increases with mean return and decreases with return volatility
- Modeled via utility function

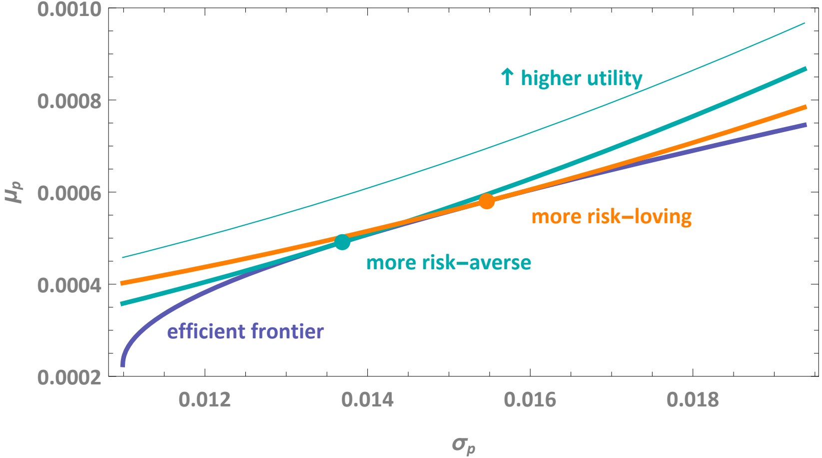
$$V(\mu_p, \sigma_p) = \mu_p - \frac{1}{2} k \sigma_p^2,$$

  with k expressing strength of investor's risk aversion
- **Indifference curves** express mean/volatility tradeoff
  - Defined by fixing utility at V° and differentiating the utility function

$$\left.\frac{d\mu_p}{d\sigma_p}\right|_{V=V^\circ} = k\sigma_p$$

  - The slope is positive: investor must be compensated with additional expected return if risk increases
  - Convex to the origin: slope is increasing in σ_p
- Investor chooses efficient portfolio that just touches the highest indifference curve she can achieve

Optimal investor choice among portfolios



Indifference curves for utility function $V(\mu_p, \sigma_p) = \mu_p - \frac{1}{2}k\sigma_p^2$ with $k = 4$ and $k = 2$ and efficient frontier of portfolios combining long positions in KO stock and FB stock.

Investor choice if there is a risk-free asset

- Suppose there really were a risk-free security with certain return r^f
  - Its mean would also be r^f and its volatility zero
- Suppose investor able to lend or borrow freely at risk-free rate
  - Lending: invest in risk-free asset
  - Borrowing: finance additional risky assets (→leverage)
- We can then define

Expected excess return of an asset: the difference $\mu_i - r^f$ between its expected return and the risk-free rate

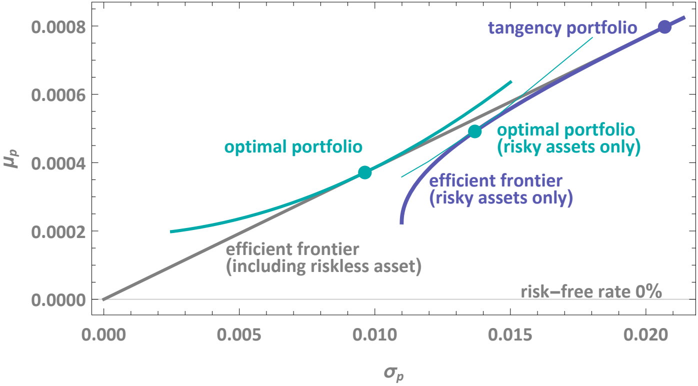
Sharpe ratio of an asset: ratio $\frac{\mu_i - r^f}{\sigma_i}$ of excess return to volatility

- Expected excess return per unit of risk
- Reported Sharpe ratios usually *ex post*, based on realized/historical estimate of expected future return

Two-fund separation

- Also called **mutual fund theorem**
- All investors have same risky asset portfolio but different amounts of risk-free asset and risky portfolio
  - Risky asset portfolio has same constituents and same weights within the portfolio for everyone
- What is that risky asset portfolio?
- If there is a risk-free asset, efficient frontier→ray from $(0, r^f)$ through **tangency portfolio**
  - Tangency portfolio is risky asset portfolio common to all investors
  - Tangency⇒attainable risky asset portfolio with highest Sharpe ratio
  - ⇒Slope of efficient frontier is highest attainable Sharpe ratio
- Investor mixes risk-free asset and risky portfolio
  - The mix depends on her risk preferences

Optimal investor choice with a risk-free asset



Efficient frontier of portfolios combining only long positions in KO stock and FB stock, efficient frontier of portfolios that also include a risk-free asset, with $r^f = 0$, and the indifference curve for $V(\mu_p, \sigma_p) = \mu_p - \frac{1}{2}k\sigma_p^2$ with $k = 4$ at the optimal portfolio of a risk-free as well as risky assets. The **tangency portfolio** is the point on the efficient frontier of portfolios combining risky assets only that is tangent to the efficient frontier, a line through $(0, r^f)$.

Summary of optimal investor choice

Weight	No risk-free asset		Including risk-free asset		
	$k=3$	$k=4$	tangency	$k=3$	$k=4$
Risk-free	NA	NA	NA	0.379	0.535
FB	0.586	0.472	0.866	0.537	0.403
KO	0.414	0.528	0.134	0.083	0.063

Weights in the optimal portfolio of a mean-variance investor with utility function $V(\mu_p, \sigma_p) = \mu_p - \frac{1}{2}k\sigma_p^2$ for $k = 3, 4$. Optimization is over portfolios combining only long positions in KO stock and FB stock, or portfolios that also include a risk-free asset.

Risk premiums and equilibrium market prices

- Risky asset prices embed **risk premiums**:
    - *Expected* excess return $\mu_i - r^f$ of risky (r_i) over risk-free security
    - Market discounts risky income streams at risk-free rate *plus* risk premium
- Investment i's Sharpe ratio $\frac{\mu_i - r^f}{\sigma_i}$ is the ratio of risk premium to volatility
- Risk premiums not directly observable, must be estimated via model of how market finds equilibrium asset prices
    - Equilibrium prices then related to investor preferences as well as assets' return characteristics
    - Leads also to explanation of relative prices of different risky assets

Overview

- **Capital asset pricing model** (CAPM):
  - All risk premiums driven by risk appetites and common source of risk
- Risk premium of any security is compensation for **systematic risk**
  - Related to risk of **value-weighted market portfolio** of all risky assets
  - Obviates need for or validity of security-specific analysis (→efficient markets)
- Diversification→shedding *uncompensated* **nonsystematic** or **idiosyncratic risk**
- CAPM a model of relative rather than absolute risk
  - CAPM does not itself provide measure of risk of market portfolio, i.e. systemic risk
  - ⇒Volatility estimation

CAPM assumptions

- Agents are all mean-variance optimizers, and don't care about other distributional characteristics
- Complete information and agreement on means and variances of uncertain security returns
- Agents are not, however, identical
  - But they may have different risk preferences/aversion:⇔differ in their pricing of risk
- Market portfolio well-defined, has identifiable observable counterpart, non-traded assets unimportant
  - Conventionally proxied by broad stock index, e.g. S&P 500, the observable **market factor**
- Market clearing with no frictions
- There is a risk-free asset at which all agents can freely borrow or lend
  - Risk-free rate typically proxied by U.S. Treasury bill yield or return

Key results of the CAPM

- The market portfolio is an efficient portfolio
  - Since all investors agree on expected return and volatility of each asset, each chooses the same portfolio of risky assets
  - All markets clear, so all investors must be choosing market portfolio to combine with risk-free asset
- **Mutual fund theorem:** all investors will engage in two-fund separation
  - Each will choose a mix of the market portfolio and the risk-free asset depending on her own risk preferences
- The market portfolio is the only source of risk
  - CAPM consistent with single risk premium "risk-on/risk-off" world

The CAPM beta

- CAPM a model of prices and risk relative to market portfolio
  - Any specific asset *i*'s risk premium related via **beta** to co-movement with that of market portfolio $\mu_m - r^f$

$$\mu_i - r^f = \beta_i(\mu_m - r^f)$$

  - Specific securities thereby priced relative to one another
- Where's α? It's *zero* in the CAPM
- An asset *i*'s beta can be calculated from its excess return volatility σ_i, that of the market portfolio σ_m, and their correlation $\rho_{i,m}$:

$$\beta_i = \rho_{i,m}\frac{\sigma_i}{\sigma_m}$$

  - Beta increasing in correlation *and* asset volatility

The CAPM and risk premiums in equilibrium

- Any asset's Sharpe ratio related to that of the market portfolio by

$$\frac{\mu_i - r^f}{\sigma_i} = \rho_{i,m} \frac{\mu_m - r^f}{\sigma_m}$$

- Since $\rho \leq 1$, no asset can have a higher Sharpe ratio in equilibrium than the market portfolio

Systematic and nonsystematic risk

- Asset i's excess return variance σ_i^2 a measure of its risk
- Market portfolio's excess return variance σ_m^2 a measure of market risk
    - Beta captures comovement with/risk sensitivity to market factor
- σ_i^2 can be decomposed into

  Systematic risk: $\beta_i^2 \sigma_m^2$, the part of σ_i^2 (or σ_i) related to fluctuations in market returns

  Idiosyncratic or **nonsystematic risk:** the remainder $\sigma_i^2 - \beta_i^2 \sigma_m^2$, due to vagaries of individual firm's returns alone
- Can be expressed as shares of total

$$\frac{\beta_i^2 \sigma_m^2}{\sigma_i^2} + \frac{\sigma_i^2 - \beta_i^2 \sigma_m^2}{\sigma_i^2} = 1$$

- Systematic risk share in terms of excess return correlation:

$$\frac{\beta_i^2 \sigma_m^2}{\sigma_i^2} = \rho_{i,m}^2 \frac{\sigma_i^2}{\sigma_m^2} \frac{\sigma_m^2}{\sigma_i^2} = \rho_{i,m}^2$$

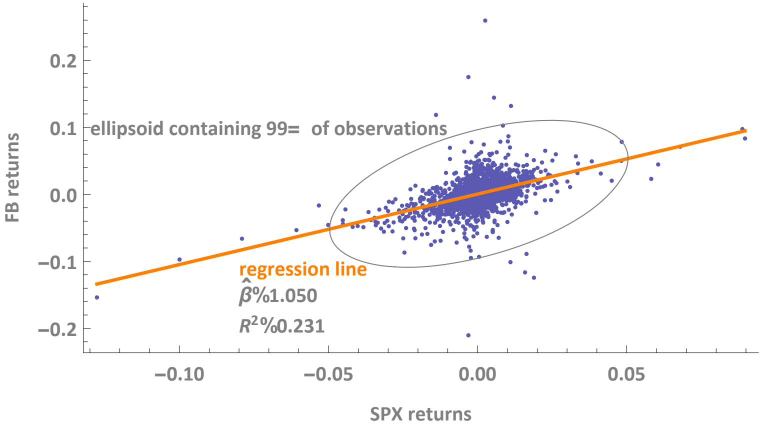
Computing the CAPM beta

- Based on simple **linear regression** model of security i's excess returns on market portfolio's excess return

$$r_{it} - r_{ft} = \alpha_i + \beta_i(r_{mt} - r_{ft}) + u_{it}, \qquad t = 1, \ldots, T$$

  - u_{it} assumed i.i.d. or normal, and independent of $r_{mt} - r_{ft}$
  - Daily, weekly or monthly observations
- Leads to estimates $\hat{\alpha}_i$ and $\hat{\beta}_i$
  - The CAPM model predicts $\alpha_i = 0$
- $\hat{\sigma}_{u,i}$ is the **residual mean square** or **standard error of the regression**
  - $\hat{\sigma}^2_{u,i}$ an unbiased estimate of variance $\sigma^2_{u,i}$ of the model error term u_i

Computing beta: an example



Computation of beta of the FB to S&P 500, using 2087 unweighted daily excess return observations 18May2012 to 24Sep2020, relative to 3-month U.S. T-bill yield at the beginning of the return period. Points mark daily excess return pairs, expressed as decimals.

Estimating systematic and nonsystematic risk

- Interpret standard regression properties in context of CAPM
- Decompose stock i's observed excess return variance $\frac{\sum_t (r_{it} - r_{ft})^2}{T-1} = \hat{\sigma}_i^2$ into
  - Explained and residual variance
  - $\rightarrow$Systematic and nonsystematic risk
- Explained or systematic variance can be expressed as

$$\hat{\beta}_i^2 \frac{\sum_t (r_{mt} - r_{ft})^2}{T-1} = \hat{\beta}_i^2 \hat{\sigma}_m^2 = R^2 \hat{\sigma}_i^2$$

  - The estimated $\hat{\beta}_i^2$ times the variation in market excess returns
  - The estimated **coefficient of determination** (unadjusted) R^2 times the variation in stock i's excess returns
  - R^2 equals sample excess return correlation
- Residual or nonsystematic variance is the difference:

$$\hat{\sigma}_i^2 - \hat{\beta}_i^2 \hat{\sigma}_m^2 = \frac{T-2}{T-1} \hat{\sigma}_{u,i}^2$$

Example: systematic and nonsystematic risk

	FB	KO	KBH
Parameter estimates:			
$\hat{\alpha}_i$	0.0004435	-0.0001763	0.0001699
$\hat{\beta}_i$	1.051	0.674	1.456
Goodness-of-fit and correlation:			
R^2 (unadjusted)	0.23068	0.40096	0.28040
Adjusted R^2	0.23032	0.40067	0.28006
Excess return correlation to S&P $\hat{\rho}_{i,m}$	0.21293	0.22568	0.48030
Risk decomposition:			
Variance of excess returns $\hat{\sigma}_i^2$	0.0005507	0.0001302	0.0008689
Systematic variance $\hat{\beta}_i^2\hat{\sigma}_m^2$	0.0001270	0.0000522	0.0002436
Nonsystematic variance $\hat{\sigma}_i^2 - \hat{\beta}_i^2\hat{\sigma}_m^2$	0.0004236	0.0000780	0.0006253
Risk decomposition (share of total variance):			
Systematic variance	0.23068	0.40096	0.28040
Nonsystematic variance	0.76932	0.59904	0.71960

Excess returns relative to 3-month U.S. T-bill yield.

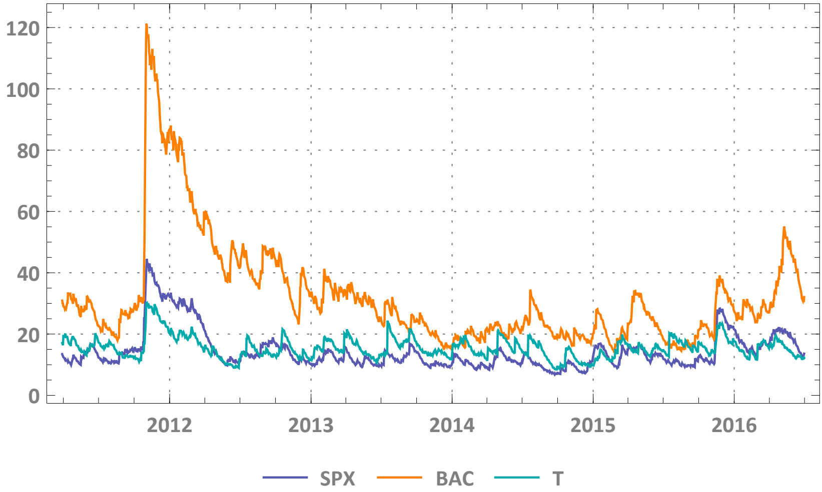
Beta, correlation, and volatility

- Excess returns of high-beta stocks typically, but not always strongly correlated with market's
- Relationships among beta, correlation, and market and asset volatilities constrained by

$$-1 \leq \beta_i \frac{\sigma_m}{\sigma_i} = \rho_{i,m} \leq 1$$

- If high-beta returns much more volatile than market returns, correlation may be weak
  - $\Leftrightarrow$Systematic risk share low, in spite of high beta
- High beta associated with higher return volatility than market, but tracking overall behavior of market return volatility
- **Examples:**
  - BAC has beta nearly triple that of T, but only moderately higher excess return correlation to the market
  - BAC has high beta, but systematic risk just a bit over $\frac{1}{2}$ of total

Volatility behavior of high- and low-beta stocks



Annualized EWMA volatility for BAC, T and the S&P 500, daily 03Jan2011 to 30Jun2016.

Empirical validation of the CAPM

- CAPM implies **cross-sectional variation** across individual stocks in *expected* returns at a point in time fully explained by beta
    - Expected returns measured empirically via realized excess returns
- In tests, CAPM does not fully capture systematic influences on individual stock prices
- CAPM a **single-factor** or **single-index** model
    - ⇒Search for additional explanatory factors
- **Fama-French three-factor model** includes in addition to market factor

  Small Minus Big (SMB): average return on small-cap minus return on large-cap portfolios

  High Minus Low (HML): average return on value (high **book-to-market**/low **price-to-book ratio**) minus return on growth (low book-to-market) portfolios
- **Momentum** factor: stocks with high recent returns

Limitations of the market proxy

- **Roll critique** or **market proxy problem**
- Validation of CAPM requires accurate identification of market portfolio
- Conventional proxies, e.g. S&P 500 index omit important elements of wealth, esp. human capital, non-U.S. assets
- Analogous to the (→)**joint hypothesis problem** in testing market efficiency
  - Is the model or the proxy wrong?

Consumption CAPM

- Simplicity of the mean-variance optimization model contributes to empirical shortcomings of CAPM
- Investors care about many things, e.g.
  - Do high payoffs occur in good times or in bad, when they are more valuable?
  - Do high payoffs occur when investment opportunities are good, or capital goods cheap relative to consumption goods?
  - Tail risks, **rare consumption disasters**, e.g. financial crises and wars
- **Consumption CAPM:** asset prices driven by risk appetite, covariance of return with utility of consumption across states
  - Multiple periods, not just "now" and "future"
  - Declining **marginal utility of consumption:** additional consumption less valuable at higher consumption level
  - ⇒Low asset payoffs in bad times less valuable to risk-averse agents ("anti-insurance"), lead to lower asset price/higher risk premium
  - ⇔High payoffs in bad times→higher asset price/lower risk premium

Stochastic discount factor and asset prices

- **Stochastic discount factor:** (SDF) discounted value of marginal utility of consumption, captures
  - **Time preference:** near-term more valuable than future consumption
  - **Risk preference:** how fast does marginal utility of consumption decline?
- SDF the same (for a particular or representative agent) for all assets
  - A different value of the SDF for each possible future state
- But state-contingent payoffs different for each asset
- Assets have positive risk premiums—are cheaper—if covariance of payoffs with SDF high

Risk factor approach to asset pricing

- Reduce dimensionality of covariance matrix of asset returns
  - Economic/market data that explain *variance* of returns
  - Possibly unobservable or **latent** characteristics
  - Definition of risk factors depends on model and available data
  - Asset returns as risk factors for other securities
- In efficient capital markets, risk premiums reflect **priced factors**
- $\rightarrow$**Arbitrage pricing theory** (APT) introduces multiple risk factors
  - Asset or portfolio returns accurately predicted by returns on a set of factors$\Rightarrow$portfolio can be replicated by the factors
  - **Example**: Fama-French model prices stocks more accurately than CAPM
  - Each factor carries with it a risk premium that compensates for low return just when you can least afford it
- Two-fold motivation of risk factor approach:

  Reality: many securities, far fewer meaningfully independent influences on them

  Parsimony: make high-dimensional problem tractable and intuitive