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UNIFORM LARGE DEVIATIONS FOR HEAVY-TAILED QUEUES UNDER HEAVY TRAFFIC

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ABSTRACT. We provide a complete large and moderate deviations asymptotic for the steady-state waiting time of a class of subexponential $M/G/1$ queues under heavy traffic. The asymptotic is uniform over the positive axis, and reduces to heavy-traffic asymptotics and heavy-tail asymptotics on two ends, both of which are known to be valid over restricted asymptotic regimes. The link between these two well-known asymptotics is a transition term that is expressible as a convolution-type integral. The class of service times that we consider includes regularly varying and Weibull tails in particular.

It is our pleasure to contribute to this special issue dedicated to the International Year of Statistics. In response to the request of the editors of this special issue we briefly overview the research topics that we have investigated recently. Our research group has pursued several themes in recent years. All of them lie under the scope of applied probability. Some of our projects deal with computational probability. In this context, our goal is to enable efficient computation in stochastic systems using (and often developing) theory of probability to inform the design of algorithm that are optimal and robust in certain sense (see Blanchet and Glynn (2008)). Most of the computations that we study relate to stochastic simulation (also known as Monte Carlo) methods (see Blanchet and Lam (2012)). Other projects that we pursue relate to classical analysis in probability, such as asymptotic approximations, large deviations, and heavy-traffic limits (Blanchet and Glynn (2006) and Lam et al (2011)). All of our research efforts are motivated by models and problems in areas such as: Finance, Insurance, Operations Research, and Statistics.

Here we shall study a class of asymptotic results that lie at the intersection of large deviations and heavy-traffic limit theory. We use a classical model in queueing theory to illustrate these types of results, namely, the classical $M/G/1$ queue. Despite its apparent tractability, most of the *asymptotics for the steady-state waiting time of the $M/G/1$ queue* that have been proposed in the literature are only provably valid in restricted regimes. Among them are the well-known heavy-traffic or Kingman asymptotic (see Kingman (1961)) and the heavy-tail or Pakes-Veraberbeke asymptotic (see for example Embrechts and Veraverbeke (1982)). More precisely, in heavy traffic (i.e. when the long-run proportion of time the server is utilized, ρ, is close to 1) one approximates the distribution of the steady-state waiting time in spatial scales of size $1/(1-\rho)$ by the steady-state distribution of reflected Brownian motion (which is exponential). On the other hand, the heavy-tail asymptotic assumes fixed traffic intensity while the tail parameter

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increases. It states that for service time with so-called stationary excess distribution (in the tradition of renewal theory) $B_0(x)$ lying in the class, $\mathcal{S}$, of subexponential distribution (see, for example, Embrechts et. al. (2003) and Asmussen (2001, 2003)), the probability that the steady-state waiting time is larger than x is asymptotically $(\rho/(1-\rho))\bar{B}_0(x)$.

In this paper we provide a uniform large deviations asymptotic of the steady-state waiting time distribution for heavy-tailed $M/G/1$ under heavy traffic. Our results can handle the case when heavy traffic is present but the tail parameter level is moderate, which is covered by neither heavy-traffic or heavy-tail asymptotics. Our results in this paper extend and unify previous work by Olvera-Cravioto et. al. (2011) and Olvera-Cravioto and Glynn (2011). In these two papers, the authors studied first the regularly varying $M/G/1$ queue and showed that heavy-traffic and heavy-tail asymptotics remain valid on regimes that are respectively smaller and larger than an explicitly identified transition point. Then, a separate argument is given in Olvera-Cravioto and Glynn (2011) in order to deal with Weibull type distributions. Our framework here provides means to develop a unified theory of transitions from heavy-traffic to heavy-tailed asymptotics that covers both regularly varying and Weibullian tails at once. In addition, and in contrast to Olvera-Cravioto et. al. (2011), for regularly varying distributions we provide an explicit asymptote for the behavior of the tail of the steady-state waiting time in the $M/G/1$ queue right at the transition point in complete generality.

We shall use the machinery developed by Rozovskii (1989, 1993) for large and moderate deviations of random walks. Related papers that develop similar methods include Nagaev (1969) and Borokov and Borovkov (2001). A central argument to obtaining these deviations results is finding a suitable truncation level depending on p. In Section 2 we outline this truncation argument and we provide the details of the proofs in Section 3.

1. Statement of Result and Outline of Argument

Let $(X_i : i \geq 1)$ be a sequence of non-negative i.i.d. r.v.'s (independent and identically distributed random variables) and define $S_n = X_1 + \cdots + X_n$, with $S_0 = 0$. Let M be a geometrically distributed random variable with parameter $p > 0$ and independent of the X_i's. In particular, $P(M = k) = pq^k$ for $k = 0, 1, ...,$ where $q = 1 - p$. The random variable S_M is said to be a geometric sum. It turns out (see Asmussen (2003)), that the steady-state waiting time of an $M/G/1$ queue can be represented as geometric sum, is $p = 1 - \rho > 0$.

Throughout the rest of the paper we use $F(x)$ to denote the distribution of X_i and we write X to denote a generic copy of X_i. We are interested in $P(S_M > x)$ as $p \searrow 0$ and $x = x(p) \nearrow \infty$.

We use the following notation. Given non-negative functions $f_1(\cdot)$, $f_2(\cdot)$, we write $f_1(x) \ll f_2(x)$ if $f_1(x) = o(f_2(x))$ as $x \to \infty$ (i.e. "f_1 has smaller order than f_2") and $f_1(x) \gg f_2(x)$ if $f_2(x) = o(f_1(x))$ (i.e. "f_1 has larger order than f_2"). Also we use "$\preceq$" and "$\succeq$" to denote "having order smaller than or equal to" and "having order larger than or equal to", respectively. For example, $f_1(x) \preceq f_2(x)$ means that $f_1(x) \leq cf_2(x)$ for some $c > 0$. Lastly, we use "$\sim$" to denote "asymptotically equivalent or the same order" (i.e. $f_1(x)/f_2(x) \to 1$).

We shall consider X_i in class $\mathcal{S}$ of subexponential distribution (i.e. $P(X_1+X_2 > x)/\bar{F}(x) \to 2$) together with the assumption

$$\bar{F}(x) := P(X_i > x) = e^{-g(x)},$$

where $g(\cdot) \geq 0$ is clearly non-decreasing, and $g(x) \to \infty$ as $x \to \infty$. We also assume that $g(\cdot)$ is differentiable, and that $g(x)/x^{\delta} \to 0$ and is eventually decreasing for some $0 < \delta < 1$. We further assume that $EX = \mu < \infty$, $EX^2 = 1$ and $EX^{2+\epsilon} < \infty$ for some $\epsilon > 0$ (we sometimes drop the subscript i in X_i for convenience). We assume that $(2+\epsilon)\log x \leq g(x) \ll x$. In addition, we assume that $h(x) = g(x+\mu) - 2\log x$ is eventually non-decreasing and goes to ∞, which is intuitive given that $EX^{2+\epsilon} < \infty$. We also assume that $h'(x)/h(x) \leq (\delta+\eta)/x$ for some $\eta > 0$ eventually. Finally, we also assume that X_i is strongly non-lattice, in the sense that

$$\inf_{|\omega|>v} |1-\chi(\omega)| > 0$$

for any $v > 0$, where $\chi(\omega) = Ee^{i\omega X}$ is the characteristic function of X.

Set $B_p = 1/p$ and define for $x > 0$ and $p \in (0,1)$

$$\Gamma(x,p) = \left[e^{-\theta^* x} + \left(\frac{1}{p}\bar{F}(x) + \int_{B_p}^{x} \left(\frac{1}{p} + \frac{x-y}{\mu} \right) e^{-\theta^*(x-y)} dF(y) \right) I(x \geq B_p) \right], \tag{1.1}$$

where θ^* is the solution to the equation $E[e^{\theta X}; X \leq B_p] = 1/q$. Our main result is the following:

THEOREM (1.2). *Let $B_p = 1/p$. We have uniformly over $x > 0$ that*

$$\lim_{p\to 0} \sup_{x>0} \left| \frac{P(S_M > x)}{\Gamma(x,p)} - 1 \right| = 0.$$

Note that we can as well choose B_p to be any quantity having the same order as $1/p$. Blanchet and Glynn (2007) shows that θ^* admits a Taylor series type expansion $\theta^* = p/\mu + c_2 p^2 + \cdots$. The expansion is valid up to the order of the moment of X_i. Thus if $EX^2 < \infty$ we can ensure that θ^* can be expanded up to the second order of p. Note that this gives $e^{-\theta^* x} \sim e^{-px/\mu}$ for $B_p \leq x \ll 1/p^2$, which coincides with Kingman's asymptotic. On the other hand, the second term in (1.1) is the heavy-tail asymptotic. It can be shown that the first term is dominant for small order of x (with respect to p) while the second term is dominant for large order. The third term can be the dominant component in a neighborhood of the transition between the first and the second. These observations are apparent through the following example.

EXAMPLE (1.3) (Regularly Varying Tail). *Suppose X_i has density $L(x)/x^{1+\alpha}, x > 0$ where $\alpha > 2$ and $L(\cdot)$ is a slowly varying function, so $\bar{F}(x) \sim L(x)/x^{\alpha}$. We are interested in computing the third term of (1.1). First we have*

$$\frac{1}{p}\int_{B_p}^{x} e^{-\theta^*(x-y)} \frac{L(y)}{y^{1+\alpha}} dy = \frac{1}{p} e^{-\theta^* x} \frac{L(x)}{x^{\alpha}} \int_{B_p/x}^{1} e^{\theta^* xu} \frac{1}{u^{1+\alpha}} \frac{L(ux)}{L(x)} du$$

$$\sim \frac{1}{p} e^{-\theta^* x} \frac{L(x)}{x^{\alpha}} \int_{B_p/x}^{1} \frac{e^{\theta^* xu}}{u^{1+\alpha}} du.$$

where the first equality follows by a substitution $y = xu$, and the equivalence relation follows from the property of slowly varying function that $L(ux)/L(x) \to 1$ uniformly over compact set as $x \to \infty$. If $\theta^* x = O(1)$, which implies $x \le C_1/\theta^* \sim C_1\mu/p$ for some constant $C_1 > 0$ (see the proof of Lemma (2.4) for the equivalence $\theta^* \sim p/\mu$), then

$$\int_{B_p/x}^{1} \frac{e^{\theta^* xu}}{u^{1+\alpha}} du \le e^{\theta^* x} \int_{B_p/x}^{1} \frac{1}{u^{1+\alpha}} du \le C_2 e^{\theta^* x}$$

for some $C_2 > 0$ and so

$$\frac{1}{p} e^{-\theta^* x} \frac{L(x)}{x^\alpha} \int_{B_p/x}^{1} \frac{e^{\theta^* xu}}{u^{1+\alpha}} du \le \frac{C_2}{p} \frac{L(x)}{x^\alpha} \ll e^{-\theta^* x}.$$

On the other hand, if $\theta^* x \nearrow \infty$, then applying Laplace's method yields

$$\frac{1}{p} e^{-\theta^* x} \frac{L(x)}{x^\alpha} \int_{B_p/x}^{1} \frac{e^{\theta^* xu}}{u^{1+\alpha}} du \le \frac{1}{p} \frac{L(x)}{x^\alpha} \frac{1}{\theta^* x} \ll \frac{1}{p} \bar{F}(x).$$

Now by the same analysis, and noting that $\theta^* \sim p/\mu$, we obtain that

$$\frac{x}{\mu} \int_{B_p}^{x} e^{-\theta^*(x-y)} \frac{L(y)}{y^{1+\alpha}} dy \le C_3 \frac{L(x)}{x^{\alpha-1}} \ll e^{-\theta^* x}$$

and

$$\frac{1}{\mu} \int_{B_p}^{x} e^{-\theta^*(x-y)} \frac{L(y)}{y^{\alpha}} dy \le C_4 \frac{L(x)}{x^{\alpha-1}} \ll e^{-\theta^* x}$$

for $\theta^* x = O(1)$, and

$$\frac{x}{\mu} \int_{B_p}^{x} e^{-\theta^*(x-y)} \frac{L(y)}{y^{1+\alpha}} dy \sim \frac{1}{\mu} \int_{B_p}^{x} e^{-\theta^*(x-y)} \frac{L(y)}{y^{\alpha}} dy \sim \frac{1}{p} \frac{L(x)}{x^\alpha}$$

for $\theta^* x \nearrow \infty$, which implies that

$$\int_{B_p}^{x} \frac{x-y}{\mu} e^{-\theta^*(x-y)} \frac{L(y)}{y^{1+\alpha}} dy = o\left(\frac{1}{p} \frac{L(x)}{x^\alpha}\right).$$

Hence we have

$$P(S_M > x) = \left[e^{-\theta^* x} + \frac{1}{p} \bar{F}(x)\right](1 + o(1))$$

for any $x > 0$. This recovers a basic result in Olvera-Cravioto et. al. (2011) and identifies the transition point located at $-((\alpha-2)/2)\log(p)/p$.

We now give a brief outline of our argument leading to Theorem (1.2). Detailed proofs will be provided in the next section. Our method is mainly inspired by Rozovskii (1989, 1993) together with the use of uniform renewal theorem in Blanchet and Glynn (2007). We first find an appropriate truncation for X_i, so that the geometric sum of the truncated part can be approximated by uniform renewal theorem while the remaining part follows the big-jump asymptotic. Uniform renewal theory then yields Kingman's asymptotic. On the other hand, the heavy-tail component will boil down to calculating a convolution of negative binomial sum with the increment distribution.

From now on we will adopt the following notations. Recall that $B_p = 1/p$, and δ satisfies $g(x)/x^\delta \to 0$. This allows us to find $\delta' = \delta + \eta < 1$ for some $\eta > 0$. For convenience of development, when $g(x) \preceq \log x$, we take $\delta = 0$ and δ' be a small number such that $0 < \delta' < 1$. We then let $K_p = (1/p^{2\delta})e^{(1-\delta')g(B_p)}$, and

$$C_{p,M} = \begin{cases} B_p & \text{for } M \le K_p \\ \mu + \sqrt{M} & \text{for } M > K_p \end{cases}$$

Let us state the result on the split into truncated and remaining part:

PROPOSITION (1.4).

$$\begin{aligned} P(S_M > x) \quad = \quad & \left[P\left(S_M > x, \max_{1 \le i \le n} X_i \le C_{p,M}\right)\right. \\ & \left. + P\left(S_M > x, \bigcup_{i=1}^{n} \left\{X_i > C_{p,M}, \max_{j \ne i} X_j \le C_{p,M}\right\}\right)\right](1 + o(1)) \end{aligned} \tag{1.5}$$

uniformly over $x > 0$.

Note that we have used a truncation level that remains at B_p for small M but grows in order $\sqrt{M}$ for large M. Such level will ensure that the contribution of two or more jumps i.e. $X_i > C_{p,M}$, is negligible for both small and large M. Moreover, as we shall see in Proposition (1.6) below, K_p is chosen such that the truncated part is regular enough to invoke uniform renewal theorem.

Our argument is finished by recognizing the two components in the right hand side of (1.5) as the terms in (1.1), via the following propositions:

PROPOSITION (1.6).

$$P\left(S_M > x, \max_{1 \le i \le n} X_i \le C_{p,M}\right) = e^{-\theta^* x} + o\left(e^{-\theta^* x} + \frac{1}{p}\bar{F}(x) I(x \ge B_p)\right)$$

uniformly over $x > z(p)$ for any $z(p)$ such that $z(p) \to \infty$ as $p \to 0$.

PROPOSITION (1.7).

$$P\left(S_M > x, \bigcup_{i=1}^{n} \left\{X_i > C_{p,M}, \max_{j \ne i} X_j \le C_{p,M}\right\}\right)$$
$$= \begin{cases} \left[\frac{1}{p}\bar{F}(x) + \int_{B_p}^{x}\left(\frac{1}{p} + \frac{x-y}{\mu}\right)e^{-\theta^*(x-y)} dF(y)\right](1 + o(1)) & \textit{uniformly over } x \ge B_p \\ \ll e^{-\theta^* x} & \textit{uniformly over } x < B_p \end{cases}$$

2. Proofs

Note that Theorem (1.2) is uniform over $x \ge 0$. The results that follow are obtained uniformly over $x \ge z(p)$ as long as $z(p) \to \infty$ as $p \to 0$. Of course, for $x \le z(p) < B_p$ we have that $\Gamma(x,p) = e^{-\theta^* x}$ and therefore the limit

$$\lim_{p \to 0} \sup_{0 \le xp \le z(p)p} \left| \frac{P(pS_M > px)}{\Gamma(x,p)} - 1 \right| = \lim_{p \to 0} \sup_{0 \le u \le z(p)p} \left| \frac{P(pS_M > u)}{\exp(-(\theta^*/p)u)} - 1 \right| = 0,$$

is easily established if $z(p)p = o(1)$. Consequently, to obtain Theorem (1.2) it suffices to indeed assume $x \ge z(p)$ as indicated.

Proof of Proposition (1.4). First we write

$$\begin{aligned} P(S_M > x) &= P(S_M > x,\ \max X_i \le B_p,\ M \le K_p) \\ &\quad + P(S_M > x,\ \max X_i \le \mu + \sqrt{M},\ M > K_p) \\ &\quad + P(S_M > x,\ \max X_i > B_p,\ M \le K_p) \\ &\quad + P(S_M > x,\ \max X_i > \mu + \sqrt{M},\ M > K_p) \end{aligned} \tag{2.1}$$

Note that the first two terms constitute the first term in the right hand side of (1.5), and we shall focus on the last two terms. For the third term, we have

$$\begin{aligned} &P(S_M > x,\ \max X_i > B_p,\ M \le K_p) \\ &\quad = P(S_M > x,\ \text{exactly one } X_i > B_p,\ M \le K_p) \\ &\qquad + P(S_M > x,\ \text{more than one } X_i > B_p,\ M \le K_p). \end{aligned} \tag{2.2}$$

We will show that the second term in (2.2) is negligible compared to the first term in (2.1), by following the proof in Lemma 4 of Rozovskii (1993). Using the notation there, we denote

$$Q_{n-k,k}(x) = P(S_n > x,\ X_1, X_2, \ldots, X_k > B_p,\ X_{k+1}, \ldots, X_n \le B_p),$$

$A = \sup_{y \ge 2B_p} I_{B_p}(y)/\bar{F}(y)$ where

$$I_{B_p}(y) = \int_{B_p}^{y-B_p} \bar{F}(y-u)\,dF(u),$$

and $\xi_{B_p} = \sup_{y \ge B_p} \bar{F}(y)/\bar{F}(y+B_p)$. Now Lemma 4a in Rozovskii (1993), applying to our case, states that for $k \ge 2$, $Q_{n-k,k}(x) \le AQ_{n-k+1,k-1}(x) + \bar{F}(B_p)Q_{n-k+1,k-1}(x - B_p)$ (from equation (64) there), and $Q_{n-k+1,k-1}(x - B_p) \le \xi_{B_p} Q_{n-k+1,k-1}(x)$ (from equation (65) there).

Recognizing that X_i is always non-negative, we have

$$\begin{aligned} Q_{n-k,k}(x) &\le (F(B_p))^{-1} Q_{n-k+1,k}(x) \\ &\le (F(B_p))^{-1}(AQ_{n-k+1,k-1}(x) + \bar{F}(B_p)Q_{n-k+1,k-1}(x - B_p)) \\ &\le (F(B_p))^{-1}(A + \bar{F}(B_p)\xi_{B_p})Q_{n-k+1,k-1}(x) \\ &= H_p Q_{n-k+1,k-1}(x) \end{aligned} \tag{2.3}$$

where $H_p = (F(B_p))^{-1}(A + \bar{F}(B_p)\xi_{B_p})$. Lemma 4c in Rozovskii (1993) yields that (note the slight difference in the definition of $g(x)$ between there and here. We denote $\bar{F}(x) = e^{-g(x)}$ while Rozovskii defines $\bar{F}(x) \sim e^{-g(x)}/x^2$ for the case of finite variance. Thus the $g(x)$'s differs by a term of $2\log x$. We also note that in Rozovskii (1993) $B_n = \sqrt{n}$ in the case of finite variance.)

$$A = O\left(\frac{1}{B_p^2} \max_{y \ge B_p} g(y) y^{2(1-\delta)} e^{-(1-\delta)g(y)}\right), \text{ and } \xi_{B_p} = O\left(\frac{\exp\{\delta g(B_p)\}}{B_p^{2\delta}}\right).$$

Now let $\tilde{g}(x) = g(x) - 2\log x$. Then we have

$$\begin{aligned}
A &\le p^2 \max_{y \ge B_p} (\tilde{g}(y) + 2\log y) e^{-(1-\delta)\tilde{g}(y)} \\
&\le p^2 \max_{y \ge B_p} C\tilde{g}(y) e^{-(1-\delta)\tilde{g}(y)} \quad \text{for some constant} C, \\
&\quad \text{by our assumption that } g(x) \ge (2+\epsilon)\log x \\
&= Cp^2 \tilde{g}(B_p) e^{-(1-\delta)\tilde{g}(B_p)} = Cp^{2\delta}(g(B_p) - 2\log B_p) e^{-(1-\delta)g(B_p)}
\end{aligned}$$

when p is small enough, and $\xi_{B_p} = O(p^{2\delta} e^{\delta g(B_p)})$. Hence we have

$$K_p \cdot H_p \le C(F(B_p))^{-1}(g(B_p)+1)e^{-\eta g(B_p)} \to 0$$

for some $C > 0$. So

$$\begin{aligned}
& \sum_{n=2}^{\lfloor K_p \rfloor} pq^n \sum_{k=2}^{n} \binom{n}{k} Q_{n-k,k}(x) \\
\le & \sum_{n=2}^{\lfloor K_p \rfloor} pq^n \sum_{k=2}^{n} n^k H_p^{k-1} P(S_n > x,\ X_1 > B_p,\ \max_{2 \le i \le n} X_i \le B_p) \ \text{(by iterating (2.3))} \\
\le & \sum_{n=2}^{\lfloor K_p \rfloor} pq^n n \sum_{k=2}^{n} K_p^{k-1} H_p^{k-1} P(S_n > x,\ X_1 > B_p,\ \max_{2 \le i \le n} X_i \le B_p) \\
\le & \sum_{n=2}^{\lfloor K_p \rfloor} pq^n n \frac{K_p H_p}{1 - K_p H_p} P(S_n > x,\ X_1 > B_p,\ \max_{2 \le i \le n} X_i \le B_p) \ \text{(for } p \text{ small enough)} \\
\ll & \sum_{n=2}^{\lfloor K_p \rfloor} pq^n n P(S_n > x,\ X_1 > B_p,\ \max_{2 \le i \le n} X_i \le B_p)
\end{aligned}$$

and hence

$$P(S_M > x, \text{ more than one } X_i > B_p,\ M \le K_p) \ll P(S_M > x,\ \max X_i \le B_p,\ M \le K_p)$$

uniformly over $x > 0$.

We are left to prove that the fourth term in (2.1) is equivalent in order to

$$P(S_M > x, \text{ exactly one } X_i > \mu + \sqrt{M}, M > K_p)$$

Using directly the result in Lemma 4 of Rozovskii (1993), and denoting $\tilde{X}_i = X_i - \mu$ and $\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i$ as the centered random variables, we get

$$\begin{aligned}
& P(S_M > x,\ \max X_i > \mu + \sqrt{M},\ M > K_p) = \\
= & \sum_{n=\lfloor K_p \rfloor + 1}^{\infty} pq^n P(S_n > x,\ \max X_i > \mu + \sqrt{n}) \\
= & \sum_{n=\lfloor K_p \rfloor + 1}^{\infty} pq^n P(\tilde{S}_n > x - n\mu,\ \max \tilde{X}_i > \sqrt{n}) \\
= & \sum_{n=\lfloor K_p \rfloor + 1}^{\infty} pq^n P(\tilde{S}_n > x - n\mu, \text{ exactly } 1\ \tilde{X}_i > \sqrt{n})(1 + g(x,n))
\end{aligned}$$

where the last equality follows from (61) in Rozovskii (1993) and $g(x,n) \to 0$ uniformly in x as $n \to \infty$.

Since $K_p \to \infty$ as $p \to 0$, for any ϵ, we have

$$\sum_{n=\lfloor K_p \rfloor+1}^{\infty} pq^n P(\tilde{S}_n\mu > x - n\mu,\ \text{exactly } 1\ \tilde{X}_i > \sqrt{n})g(x,n)$$
$$\leq \epsilon \sum_{n=\lfloor K_p \rfloor+1}^{\infty} pq^n P(\tilde{S}_n\mu > x - n\mu,\ \text{exactly } 1\ \tilde{X}_i > \sqrt{n})$$

when p becomes small enough. Therefore

$$P(S_M > x,\ \max X_i > \mu + \sqrt{M},\ M > K_p)$$
$$\leq (1+\epsilon)P(S_M > x, \text{exactly one } X_i > \mu + \sqrt{M},\ M > K_p)$$

for p small enough. Since ϵ is arbitrary, we conclude that

$$P(S_M > x,\ \max X_i > \mu + \sqrt{M},\ M > K_p)$$
$$\sim P(S_M > x, \text{exactly one } X_i > \mu + \sqrt{M},\ M > K_p)$$

uniformly over $x > 0$.

The proof of Proposition (1.4) is complete. □

Proof of Proposition (1.6). Note that

$$\begin{aligned} & P(S_M > x,\ \max X_i \leq C_{p,M}) = \\ = \ & P(S_M > x,\ \max X_i \leq B_p,\ M \leq K_p) + P(S_M > x,\ \max X_i \leq \mu + \sqrt{M},\ M > K_p) \\ = \ & P(S_M > x,\ \max X_i \leq B_p) - P(S_M > x,\ \max X_i \leq B_p,\ M > K_p) \\ & + P(S_M > x,\ \max X_i \leq \mu + \sqrt{M},\ M > K_p) \end{aligned}$$

The proof of the proposition will be finished by the following two lemmas:

LEMMA (2.4).

$$P(S_M > x,\ \max X_i \leq B_p) = e^{-\theta^* x}(1 + o(1))$$

uniformly over $x > z(p)$ for any $z(p)$ such that $z(p) \to \infty$ as $p \to 0$.

LEMMA (2.5).

$$P(S_M > x,\ \max X_i \leq B_p,\ M > K_p)$$
$$\leq P(S_M > x, \max X_i \leq \mu + \sqrt{M},\ M > K_p) \ll e^{-\theta^* x} + \frac{1}{p}\bar{F}(x)$$

uniformly over $x > z(p)$ for any $z(p)$ such that $z(p) \to \infty$ as $p \to 0$. □

Proof of Lemma (2.4). Let $\bar{X}$ be the truncated random variable at level B_p i.e.

$$P(\bar{X} \in B) = P(X \in B | X \leq B_p)$$

for any measurable set B. We denote $P_\theta(\cdot)$ as the exponential change of measure

$$P_\theta(\cdot) = E[e^{\theta \bar{X} - \bar{\psi}(\theta)}; \cdot\ ]$$

where $\bar{\psi}(\cdot) = \log \bar{\phi}(\cdot)$ is the logarithmic moment generating function of $\bar{X}$ and $\bar{\phi}(\theta) = Ee^{\theta \bar{X}}$ is the moment generating function of $\bar{X}$. Also denote $E_\theta[\cdot]$ as the expectation under $P_\theta(\cdot)$ and $\bar{\mu}_\theta = E_\theta \bar{X}$.

Consider the change of measure of X_i with θ satisfying

$$\bar{\phi}(\theta) = \frac{1}{qF(B_p)}$$

which gives

$$E[e^{\theta X}; X \le B_p] = \frac{1}{q} \tag{2.6}$$

This equation is similar to (33) in Blanchet and Glynn (2007), but with a different truncation level. Theirs is a truncation level x while here we use B_p regardless of x.

We want to characterize the solution of (2.6), which, as we will see, will give the θ^* in Theorem (1.2). Suppose $0 \le \theta \le Cp$ for some $C > 0$ and p small enough. Write

$$\begin{aligned} E[e^{\theta X}; X \le B_p] &= \int_0^{B_p} e^{\theta y} dF(y) \\ &= \int_0^{B_p} \left(1 + \theta y + \frac{\theta^2 y^2}{2} e^{\nu \theta y}\right) dF(y) \end{aligned}$$

for some $0 \le \nu = \nu(\theta y) \le 1$. The equation is valid by our moment assumptions on X. We then get

$$E[e^{\theta X}; X \le B_p] = F(B_p) + \theta m(B_p) + R(\theta) \tag{2.7}$$

where

$$F(B_p) = 1 - \bar{F}(B_p) = 1 - e^{-g(B_p)}$$

$$m(B_p) = \int_0^{B_p} y dF(y) = \mu(1 + o(1)) \le \mu$$

as $p \to 0$ and

$$R(\theta) = \int_0^{B_p} \frac{\theta^2 y^2}{2} e^{\nu \theta y} dF(y) \le \frac{\theta^2}{2} e^C D(B_p) \le \frac{\theta^2}{2} e^C$$

where $D(B_p) = \int_0^{B_p} y^2 dF(y)$. Equating (2.7) with $1/q = 1 + p + p^2 + \cdots$, and noting that $e^{-g(B_p)} \ll p^2$, we get $\theta^* \sim p/\mu$, which also verifies that there is a unique θ^* that indeed lies in $[0, Cp]$. Henceforth we will identify this as our θ^*.

Next we have, letting T be exponentially distributed with rate θ^* in the fourth equality below,

$$\begin{aligned} P(S_M > x, \max X_i \le B_p) &= \sum_{n=1}^{\infty} p q^n P(S_n > x, \max X_i \le B_p) \\ &= \sum_{n=1}^{\infty} p (qF(B_p))^n P(\bar{S}_n > x) = \sum_{n=1}^{\infty} p E_{\theta^*}[e^{-\theta^* \bar{S}_n}; \bar{S}_n > x] \\ &= p \sum_{n=1}^{\infty} E_{\theta^*}[x < \bar{S}_n < T] = p E_{\theta^*}\left[\sum_{n=1}^{\infty} I(x < \bar{S}_n < T))\right] \\ &= p \int_x^{\infty} \theta^* e^{-\theta^* y} (\bar{U}_{\theta^*}(y) - \bar{U}_{\theta^*}(x)) dy \\ &= p e^{-\theta^* x} \int_0^{\infty} \theta^* e^{-\theta^* y} (\bar{U}_{\theta^*}(y + x) - \bar{U}_{\theta^*}(x)) dy, \end{aligned}$$

where $\bar{U}_{\theta^*}(\cdot) = \sum_{n=1}^{\infty} P_{\theta^*}(\bar{S}_n \leq \cdot)$ is the renewal measure of $\bar{X}_i$ under the measure $P_{\theta^*}(\cdot)$.

We shall now find $\bar{U}_{\theta^*}(\cdot)$. More specifically, we will show that uniform renewal theorem, as depicted in Theorem 1, part 2 of Blanchet and Glynn (2007), is valid for $\bar{X}_i$ under the exponential family $P_{\theta^*}(\cdot)$ for all small enough p. Denote $\bar{\chi}_{\theta^*}(\omega) = E_{\theta^*} e^{i\omega\bar{X}} = qF(B_p)Ee^{(i\omega+\theta^*)\bar{X}}$ as the characteristic function of $\bar{X}$ under $P_{\theta^*}(\cdot)$. The theorem requires that such family is uniformly strongly non-lattice i.e.

$$\inf_{0\leq p\leq\kappa} \inf_{|\omega|>v} |1-\chi_{\theta^*}(\omega)| > 0$$

for small enough $\kappa > 0$ and any $v > 0$, and that $\sup_{0\leq p\leq\kappa} E_{\theta^*}\bar{X}^{2+\epsilon} < \infty$. If these conditions hold then we have (as a weaker conclusion than the stated theorem in Blanchet and Glynn (2007))

$$\sup_{0\leq p\leq\kappa} \bar{\mu}_{\theta^*}^4 \left| \bar{U}_{\theta^*}(t) - \frac{t}{\mu_{\theta^*}} - \frac{E_{\theta^*}\bar{X}^2}{2\bar{\mu}_{\theta^*}^2} \right| = o(1) \tag{2.8}$$

as $t \to \infty$.

We now check the above conditions. First note that

$$\begin{aligned}\bar{\chi}_{\theta^*}(\omega) &= qF(B_p)Ee^{i(\omega+\theta^*)\bar{X}} = qE[e^{(i\omega+\theta^*)X}; X \leq B_p] \\ &= q(E[e^{i\omega X}; X \leq B_p] + \theta^* r(\theta^*,\omega)) = q(\chi(\omega) - E[e^{i\omega X}; X > B_p] + \theta^* r(\theta^*,\omega))\end{aligned}$$

where $|r(\theta^*,\omega)| \leq E[Xe^{\nu\theta^* X}; X \leq B_p]$ for some $0 \leq \nu = \nu(\theta^* X) \leq 1$ a.s. and the second equality is valid by our moment assumptions on X. Note that

$$|E[e^{i\omega X}; X > B_p]| \leq \bar{F}(B_p), \text{ and } E[Xe^{(\nu\theta^*)X}; X \leq B_p] \leq e^C \mu,$$

for some $C > 0$. Since X is non-lattice, given any $v > 0$, we have $|1-\chi(\omega)| > 1-\zeta$ for some $0 < \zeta = \zeta(v) < 1$ for all $|\omega| > v$. So for $|\omega| > v$, we have

$$\begin{aligned}|1-\bar{\chi}_{\theta^*}(\omega)| &= |1-(1-p)(\chi(\omega) - E\left[e^{i\omega X}; X > B_p\right] + \theta^* r(\theta^*,\omega)| \\ &\geq |1-\chi(\omega)| - |E[e^{i\omega X}; X > B_p]| - \theta^*|r(\theta^*,\omega)| \\ &\quad - p(|\chi(\omega)| + |E[e^{i\omega X}; X > B_p]| + \theta^*|r(\theta^*,\omega)|) > 1-\zeta'\end{aligned}$$

for some $0 < \zeta' < 1$ and small enough p. This shows that $\bar{X}$ under the exponential family $P_{\theta^*}(\cdot)$ is uniformly strongly non-lattice. Moreover, we have

$$E_{\theta^*}\bar{X}^{2+\epsilon} \leq qE[X^{2+\epsilon}e^{\theta^* X}; X \leq B_p] \leq e^C EX^{2+\epsilon}$$

for some $C > 0$. Together with our moment assumption on X, this shows that $\sup_{0\leq p\leq\kappa} E_{\theta^*}\bar{X}^{2+\epsilon} < \infty$.

Hence we can invoke the uniform renewal theorem in Blanchet and Glynn (2007). Since $P_{\theta^*}(\bar{X} > s) = qE[e^{\theta^* X}; s < X \leq B_p] \leq qe^C\bar{F}(s)$ for $s < B_p$ and is 0 otherwise, we note that $H_1^F(t)$, $H_2^F(t)$ and $H_1^F * H_1^F(t)$ in Theorem 1, part 2 of Blanchet and Glynn (2007) all go to 0 uniformly in our exponential family as $t \to \infty$. Note also that $\bar{\mu}_{\theta^*} = qE[Xe^{\theta^* X}; X \leq B_p]$. Since $Xe^{\theta^* X}I(X \leq B_p) \leq Xe^C$ which is integrable, by dominated convergence theorem and that $\theta^* \sim p/\mu$ we have $E[Xe^{\theta^* X}; X \leq B_p] \to \mu$. Hence $\bar{\mu}_{\theta^*} = \mu + o(1)$. This concludes, from (2.8), that

$$\bar{U}_{\theta^*}(y+x) - \bar{U}_{\theta^*}(x) = \frac{y}{\bar{\mu}_{\theta^*}} + R(y,x,\theta^*)$$

where $\sup_{0\le p\le \kappa, y>0}|R(y,x,\theta^*)| \to 0$ as $x\to\infty$. So

$$pe^{-\theta^* x}\int_0^\infty \theta^* e^{-\theta^* y}(\bar U_{\theta^*}(y+x)-\bar U_{\theta^*}(x))dy$$
$$= pe^{-\theta^* x}\int_0^\infty \theta^* e^{-\theta^* y}\left(\frac{y}{\bar\mu_{\theta^*}}+R(y,x,\theta^*)\right)dy$$
$$=\frac{pe^{-\theta^* x}}{\bar\mu_{\theta^*}}\left(\frac{1}{\theta^*}+o(1)\right)\sim e^{-\theta^* x}$$

uniformly over $x>z(p)$ for any $z(p)$ such that $z(p)\to\infty$ as $p\to 0$. Lemma (2.4) is proved. □

Proof of Lemma (2.5). The first inequality holds obviously when p is small enough. Thus we will focus on the order relation. As in the proof of Proposition (1.5), we let $\tilde X_i = X_i-\mu$ and $\tilde S_n=\sum_{i=1}^n \tilde X_i$ be the centered random variables and their sum. Let $P(\tilde X > x) = e^{-g(x+\mu)}$ and recall that $h(x) = g(x+\mu)-2\log x$ satisfies $h(x)/x^{\delta'}\to 0$ and is eventually decreasing, and

$$\frac{h'(x)}{h(x)}\le\frac{\delta'}{x} \tag{2.9}$$

for large enough x. Note also that $\epsilon\log x\preceq h(x)\ll x$ by construction.

By (70) and (71) in Rozovskii (1993) (note that the $g(x)$ defined there differs from ours by a term of $2\log x$ and hence $h(x)$ here will play the role of $g(x)$ in Rozovskii's paper) we have

$$P(\tilde S_n > x-n\mu, \max \tilde X_i\le\sqrt{n})\le e^{-\beta(x-n\mu)h(\sqrt{n})/\sqrt{n}} \tag{2.10}$$

for $x\ge n\mu+\Lambda_n$, where $\Lambda_n=\alpha h(\sqrt{n})\sqrt{n}$ and $\beta<\alpha/2$, for large enough n and some constant $\alpha>0$. This will be important for our development.

We now define the following functions that will prove useful for our argument. Let $l(n)=n\mu+\Lambda_n$. We extend the domain of the function l to the positive real axis and define $l^{-1}(y)=\inf\{x: l(x)\ge y\}$, so $l^{-1}(x)=x/\mu-(\alpha/\mu)h(\sqrt{l^{-1}(x)})\sqrt{l^{-1}(x)}$. Also let $r(x)=h(x)/x$, so $1/x\ll r(x)\ll 1/x^{1-\delta'}$ as $x\nearrow\infty$. Define $r^{-1}(y)=\inf\{x: r(x)\le y\}$. We then have

$$1/y^{1/(1-\delta')}\ll r^{-1}(y)\ll 1/y \tag{2.11}$$

as $y\searrow 0$.

We shall also prove a monotone property concerning the function (whose use will become clear in the argument that follows)

$$f(n):=-\beta(x-n\mu)\frac{h(\sqrt{n})}{\sqrt{n}}+n\log q$$

We have

$$f'(n)=-\beta(x-n\mu)\left(\frac{h'(\sqrt{n})}{2n}-\frac{1}{2}\frac{h(\sqrt{n})}{n^{3/2}}\right)+\beta\mu\frac{h(\sqrt{n})}{\sqrt{n}}+\log q$$
$$=-\frac{\beta}{2}\left(\frac{x}{n}-\mu\right)\left(h'(\sqrt{n})-\frac{h(\sqrt{n})}{\sqrt{n}}\right)+\beta\mu\frac{h(\sqrt{n})}{\sqrt{n}}+\log q$$

By (2.9), we have $f'(n)\ge\beta\mu h(\sqrt{n})/\sqrt{n}+\log q\ge 0$, or $f(n)$ is increasing, when $n\le (r^{-1}(-(\log q)/(\beta\mu)))^2$.

Let $R_p = r^{-1}(-(\log q)/(\beta\mu))$. We write

$$
\begin{aligned}
& P(S_M > x,\ \max X_i \le \mu + \sqrt{M},\ M > K_p) \\
= & \sum_{n=\lfloor K_p\rfloor+1}^{\infty} pq^n P(\tilde{S}_n > x - n\mu, \max \tilde{X}_i \le \sqrt{n}) \\
= & \sum_{n=\lfloor K_p\rfloor+1}^{\lfloor R_p^2\rfloor \wedge \lfloor l^{-1}(x)\rfloor} pq^n P(\tilde{S}_n > x - n\mu, \max \tilde{X}_i \le \sqrt{n}) I(\lfloor R_p^2\rfloor \wedge \lfloor l^{-1}(x)\rfloor > \lfloor K_p\rfloor) \\
& + \sum_{n=\lfloor R_p^2\rfloor \vee \lfloor K_p\rfloor+1}^{\lfloor l^{-1}(x)\rfloor} pq^n P(\tilde{S}_n > x - n\mu, \max \tilde{X}_i \le \sqrt{n}) I(\lfloor l^{-1}(x)\rfloor > \lfloor R_p^2\rfloor \vee \lfloor K_p\rfloor) \\
& + \sum_{n=\lfloor K_p\rfloor \vee \lfloor l^{-1}(x)\rfloor+1}^{\infty} pq^n P(\tilde{S}_n > x - n\mu, \max \tilde{X}_i \le \sqrt{n}) \qquad (2.12)
\end{aligned}
$$

We will now analyze the terms one by one. Note that $n \le l^{-1}(x)$ implies $x \ge l(n)$. Hence by (2.10) and our monotone property of $f(\cdot)$ we have

$$
\begin{aligned}
& \sum_{n=\lfloor K_p\rfloor+1}^{\lfloor R_p^2\rfloor \wedge \lfloor l^{-1}(x)\rfloor} pq^n P(\tilde{S}_n > x - n\mu, \max \tilde{X}_i \le \sqrt{n}) I(\lfloor R_p^2\rfloor \wedge \lfloor l^{-1}(x)\rfloor > \lfloor K_p\rfloor) \\
\le & \sum_{n=\lfloor K_p\rfloor+1}^{\lfloor R_p^2\rfloor \wedge \lfloor l^{-1}(x)\rfloor} pq^n e^{-\beta(x-n\mu)r(\sqrt{n})} I(\lfloor R_p^2\rfloor \wedge \lfloor l^{-1}(x)\rfloor > \lfloor K_p\rfloor) \\
\le & \begin{cases} \sum_{n=\lfloor K_p\rfloor+1}^{\lfloor l^{-1}(x)\rfloor} pq^{l^{-1}(x)} \exp\left\{-\beta(x - l^{-1}(x)\mu) r(\sqrt{l^{-1}(x)})\right\} & \text{for } \lfloor l^{-1}(x)\rfloor \le \lfloor R_p^2\rfloor \\ \sum_{n=\lfloor K_p\rfloor+1}^{\lfloor R_p^2\rfloor} pq^{R_p^2} \exp\left\{-\beta(x - R_p^2\mu)\left(-\frac{\log q}{\beta\mu}\right)\right\} & \text{for } \lfloor l^{-1}(x)\rfloor > \lfloor R_p^2\rfloor \end{cases}
\end{aligned}
$$

$$
\le \begin{cases} l^{-1}(x) pq^{l^{-1}(x)} e^{-\beta\alpha h^2(\sqrt{l^{-1}(x)})} & \text{for } \lfloor l^{-1}(x)\rfloor \le \lfloor R_p^2\rfloor \\ R_p^2 p e^{(x/\mu)\log q} & \text{for } \lfloor l^{-1}(x)\rfloor > \lfloor R_p^2\rfloor \end{cases} \qquad (2.13)
$$

and for $\lfloor l^{-1}(x)\rfloor > \lfloor K_p\rfloor$. The first part of the last inequality follows by substituting $x = l(l^{-1}(x)) = l^{-1}(x) + \alpha h(\sqrt{l^{-1}(x)})\sqrt{l^{-1}(x)}$. We shall prove that in both cases they are of smaller order than $e^{-\theta^* x} + (1/p)\bar{F}(x) I(x \ge B_p)$.

Consider the first case, and suppose $K_p \le x \le R_p$. Dividing the first part of (2.13) by $e^{-\theta^* x}$ gives

$$
\exp\left\{\frac{p\alpha}{\mu} h(\sqrt{l^{-1}(x)})\sqrt{l^{-1}(x)}(1+o(1)) - \beta\alpha h^2(\sqrt{l^{-1}(x)}) + \log(l^{-1}(x)p)\right\} \qquad (2.14)
$$

by substituting $l^{-1}(x) = x/\mu - (\alpha/\mu)h(\sqrt{l^{-1}(x)})\sqrt{l^{-1}(x)}$ and using $\log q = -p(1+o(1))$. Note that $x \le R_p$ implies that $r(x) \ge -(\log q)/(\beta\mu) \ge p/(\beta\mu)$. Since $r(\sqrt{l^{-1}(x)}) \ge r(x)$, we have $h(\sqrt{l^{-1}(x)}) \gg (p/(\beta\mu))\sqrt{l^{-1}(x)}$. This gives

$$
\beta\alpha h^2(\sqrt{l^{-1}(x)}) \gg (p\alpha/\mu) h(\sqrt{l^{-1}(x)})\sqrt{l^{-1}(x)}(1+o(1)).
$$

Since $h(x) \gg \epsilon \log x$, we also have $\beta\alpha h^2(\sqrt{l^{-1}(x)}) \gg \log l^{-1}(x) \ge \log(l^{-1}(x)p)$. Thus (2.14) is equal to $\exp\{-\beta\alpha h^2(\sqrt{l^{-1}(x)})(1+o(1))\} \le \exp\{-\beta\alpha h^2(\sqrt{K_p})(1+o(1))\} = o(1)$.

Now suppose $x \gg R_p$ and $\lfloor l^{-1}(x) \rfloor \le \lfloor R_p^2 \rfloor$. Note that for any $x \gg R_p$, one can always find $a = a(p) \nearrow \infty$ arbitrarily slowly as $p \searrow 0$, such that $x \gg aR_p$. Dividing the first part of (2.13) by $(1/p)\bar{F}(x)$ gives

$$\exp\{l^{-1}(x)\log q - \beta\alpha h^2(\sqrt{l^{-1}(x)}) + \log(l^{-1}(x)p) + h(x-\mu) + 2\log x + \log p\}. \tag{2.15}$$

Note that $x \gg aR_p$ implies $r(x) \ll r(x/a) \le p/(\beta\mu)$ and hence $pl^{-1}(x) \gg h(x-\mu)$, by using $l^{-1}(x) = x/\mu - (\alpha/\mu)h(\sqrt{l^{-1}(x)})\sqrt{l^{-1}(x)}$. By substituting $y = R_p$ in (2.11) we have $paR_p \ge 1/p^{\delta'/(1-\delta')} \gg -\log p$ which implies $px/\mu \gg \log x$ for $x \gg aR_p$. Hence (2.15) is equal to $\exp\{-pl^{-1}(x)(1+o(1))\} \le \exp\{-pl^{-1}(aR_p)(1+o(1))\} = o(1)$.

We now proceed to the second part of (2.13). But $l^{-1}(x) \ge R_p^2$ implies $x \gg R_p$, since $h^2(\sqrt{l^{-1}(x)})/l^{-1}(x) \to 0$. Hence by the same argument $px/\mu \gg h(x)$ and $px/\mu \gg \log x \ge \log R_p$. Hence dividing the expression by $(1/p)\bar{F}(x)$ gives

$$\exp\left\{\frac{x}{\mu}\log q + 2\log R_p + \log p + h(x-\mu) + 2\log x + \log p\right\}$$
$$= \exp\left\{-\frac{px}{\mu}(1+o(1))\right\} \le \exp\left\{-\frac{pR_p^2}{\mu}(1+o(1))\right\} = o(1)$$

We now analyze the second term of (2.12). We have, for $\lfloor l^{-1}(x)\rfloor > \lfloor R_p^2 \rfloor \vee \lfloor K_p \rfloor$,

$$\begin{aligned}
&\sum_{n=\lfloor R_p^2\rfloor \vee \lfloor K_p\rfloor+1}^{\lfloor l^{-1}(x)\rfloor} pq^n P(\tilde{S}_n > x - n\mu, \max \tilde{X}_i \le \sqrt{n}) \\
\le\ & \sum_{n=\lfloor R_p^2\rfloor+1}^{\lfloor l^{-1}(x)\rfloor} pq^n e^{-\beta(x-n\mu)r(\sqrt{n})} \le \sum_{n=\lfloor R_p^2\rfloor+1}^{\lfloor l^{-1}(x)\rfloor} p(qe^{\beta\mu r(\sqrt{n})})^n e^{-\beta x r(\sqrt{n})} \\
\le\ & \sum_{n=\lfloor R_p^2\rfloor+1}^{\lfloor l^{-1}(x)\rfloor} p(qe^{\beta\mu\frac{-\log q}{\beta\mu}})^n e^{-\beta x r(\sqrt{l^{-1}(x)})} \le \sum_{n=\lfloor R_p^2\rfloor+1}^{\lfloor l^{-1}(x)\rfloor} pe^{-\beta x r(\sqrt{l^{-1}(x)})} \\
\le\ & l^{-1}(x)pe^{-\beta x r(\sqrt{l^{-1}(x)})}
\end{aligned} \tag{2.16}$$

where the third inequality holds because $R_p = r^{-1}(-\log q/(\beta\mu))$ and $r(x)$ is eventually decreasing. Note that $xr(\sqrt{l^{-1}(x)}) \sim \mu\sqrt{l^{-1}(x)}h(\sqrt{l^{-1}(x)}) \gg h(\sqrt{l^{-1}(x)})$. Also, since $\epsilon\log x \ll h(x) \ll x$, we have $xr\sqrt{l^{-1}(x)} \ge \sqrt{l^{-1}(x)}h(\sqrt{l^{-1}(x)}) \gg \log x$. Hence dividing (2.16) by $(1/p)\bar{F}(x)$ gives

$$\begin{aligned}
&\exp\{-\beta x r(\sqrt{l^{-1}(x)}) + \log(l^{-1}(x)p) + h(x-\mu) + 2\log x + \log p\} \\
=\ & \exp\{-\beta x r(\sqrt{l^{-1}(x)})(1+o(1))\} \le \exp\{-\beta R_p^2 r(\sqrt{l^{-1}(R_p^2)})(1+o(1))\} = o(1)
\end{aligned}$$

We now analyze the final term of (2.12). We have

$$\begin{aligned}
&\sum_{n=\lfloor K_p\rfloor \vee \lfloor l^{-1}(x)\rfloor+1}^{\infty} pq^n P(\tilde{S}_n > x - n\mu, \max \tilde{X}_i \le \sqrt{n}) \le q^{l^{-1}(x)\vee K_p+1} \\
\le\ & \begin{cases} q^{K_p/a'+1} & \text{for } l^{-1}(x) \ll K_p/a' \\ q^{l^{-1}(x)+1} & \text{for } l^{-1}(x) \ge K_p/a' \end{cases} \le \begin{cases} e^{-pK_p/a'} & \text{for } l^{-1}(x) \ll K_p/a' \\ e^{-pl^{-1}(x)} & \text{for } l^{-1}(x) \ge K_p/a' \end{cases}
\end{aligned} \tag{2.17}$$

where $a' = a'(p) \nearrow 0$ as $p \searrow 0$ at a rate that will be chosen later on.

For the first part, dividing by $e^{-\theta^* x}$ yields

$$\exp\left\{-p\frac{K_p}{a'}+\theta^* x\right\}=\exp\left\{-p\frac{K_p}{a'}(1+o(1))\right\}=o(1)$$

For the second part observe that

$$r\left(\frac{K_p}{a'}\right)=\frac{h((1/a'^{2\delta})e^{(1-\delta')g(B_p)})}{(1/a'^{2\delta})e^{(1-\delta')g(B_p)}}\leq\frac{Ca'^{1-\delta'}p^{2\delta(1-\delta')}}{e^{(1-\delta'^2 g(B_p)}}\ll p$$

for some constant $C>0$, for a suitably chosen a', since $h(x)\leq x^\delta$ eventually. We then get $K_p/a'^{-1}(p)$ which implies $r(x)\leq r(K_p/a')\ll p$ for $l^{-1}(x)\geq K_p/a'$. This gives $pl^{-1}(x)\gg h(x)$. Note that $pK_p/a'=(1/a'^{1-2\delta}e^{(1-\delta')g(B_p)}\gg -2\delta\log p+(1-\delta')g(B_p)=\log K_p$, so $pl^{-1}(x)\gg\log x$ for $x\geq K_p/a'$. Hence dividing the second part of (2.17) by $(1/p)\bar{F}(x)$ gives

$$\begin{aligned}\exp\{-pl^{-1}(x)+h(x-\mu)+2\log x+\log p\}&=\exp\{-pl^{-1}(x)(1+o(1))\}\\&\leq\exp\left\{-pl^{-1}\left(\frac{K_p}{a'}\right)(1+o(1))\right\}=o(1)\end{aligned}$$

This concludes our proof of Lemma (2.5). □

Proof of Proposition (1.7). The case for $x<B_p$ is obvious, since we have $P(S_n>x$, exactly one $X_i>C_{p,n})\leq n\bar{F}(B_p)$ and hence

$$P(S_M>x,\text{ exactly one }X_i>C_{p,M})\leq\sum_{n=1}^{\infty}pq^n n\bar{F}(B_p)=\frac{q}{p}\bar{F}(B_p)=o(1)\ll e^{-\theta^* x},$$

uniformly over $x<B_p$. Thus we shall focus on $x\geq B_p$. Note that

$$\begin{aligned}&P(S_M>x,\text{ exactly one }X_i>C_{p,M})\\
\leq&\sum_{n=1}^{\lfloor K_p\rfloor}pq^n n\left(\bar{F}(x)(F(B_p))^{n-1}+\int_{B_p}^{x}P(S_{n-1}>x-y,\max X_i\leq B_p)dF(y)\right)\\
&+\sum_{n=\lfloor K_p\rfloor+1}^{\infty}pq^n n\Big(\bar{F}(x)(F(\mu+\sqrt{n}))^{n-1}\\
&+\int_{\mu+\sqrt{n}}^{x}P(S_{n-1}>x-y,\max X_i\leq\mu+\sqrt{n})dF(y)I(x\geq\mu+\sqrt{n})\Big)\\
\leq&\frac{q}{p}\bar{F}(x)+\sum_{n=2}^{\lfloor K_p\rfloor}pq^n n\int_{B_p}^{x}P(S_{n-1}>x-y,\max X_i\leq B_p)dF(y)\\
&+\sum_{n=\lfloor K_p\rfloor+1}^{\infty}pq^n n\int_{\mu+\sqrt{n}}^{x}P(S_{n-1}>x-y,\max X_i\leq\mu+\sqrt{n})dF(y)I(x\geq\mu+\sqrt{n})\\
=&\frac{q}{p}\bar{F}(x)+\sum_{n=2}^{\infty}pq^n n\int_{B_p}^{x}P(S_{n-1}>x-y,\max X_i\leq B_p)dF(y)\\
&-\sum_{n=\lfloor K_p\rfloor+1}^{\infty}pq^n n\int_{B_p}^{x}P(S_{n-1}>x-y,\max X_i\leq B_p)dF(y)\\
&+\sum_{n=\lfloor K_p\rfloor+1}^{\infty}pq^n n\int_{\mu+\sqrt{n}}^{x}P(S_{n-1}>x-y,\max X_i\leq\mu+\sqrt{n})dF(y)I(x\geq\mu+\sqrt{n}).\end{aligned}$$

We will finish the proof by invoking the following lemmas:

LEMMA (2.18).

$$\sum_{n=2}^{\infty} pq^n n \int_{B_p}^{x} P(S_{n-1} > x-y, \max X_i \le B_p) dF(y) = \int_{B_p}^{x} \left(\frac{1}{p} + \frac{x-y}{\mu}\right) e^{-\theta^*(x-y)} dF(y)(1+o(1))$$

uniformly over $x \ge B_p$.

LEMMA (2.19). *We have $\mu + \sqrt{K_p} \ge B_p$ for p small enough, and*

$$\begin{aligned} &\sum_{n=\lfloor K_p \rfloor+1}^{\infty} pq^n n \int_{B_p}^{x} P(S_{n-1} > x-y, \max X_i \le B_p) dF(y) \\ \le\ & \sum_{n=\lfloor K_p \rfloor+1}^{\infty} pq^n n \int_{B_p}^{x} P(S_{n-1} > x-y, \max X_i \le \mu + \sqrt{n}) dF(y) \\ \ll\ & \int_{B_p}^{x} \left(\frac{1}{p} + \frac{x-y}{\mu}\right) e^{-\theta^*(x-y)} dF(y) + \frac{1}{p}\bar{F}(x) \end{aligned}$$

uniformly over $x \ge B_p$. □

Proof of Lemma (2.18). We write

$$\begin{aligned} \sum_{n=2}^{\infty} pq^n n P(S_{n-1} > x, \max X_i \le B_p) &= \frac{q}{p} \sum_{n=1}^{\infty} p^2 q^n (n+1) P(S_n > x, \max X_i \le B_p) \\ &= \frac{q}{p} P(S_N > x, \max X_i \le B_p) \end{aligned}$$

where N is a negative binomial variable with parameter 2 and p. Let $\{X_i'\}_{i=1,2,\ldots}$, M' and S_M' be independent and identical copies of $\{X_i\}_{i=1,2,\ldots}$, M and S_M, and let $F_M(x) = P(S_M \le x, \max X_i \le B_p)$ and $\bar{F}_M(x)$ be its complement defined by $P(S_M > x, \max X_i \le B_p)$. Note that by Lemma (2.4) we have $P(S_M > x, \max X_i \le B_p) = e^{-\theta^* x}(1 + u(x,p))$ where $\sup_{x > B_p} |u(x,p)| \to 0$. We have

$$\begin{aligned} & P(S_N > x, \max X_i \le B_p) \\ =\ & P(S_M + S_M' > x, \max_{1\le i\le M} X_i \le B_p, \max_{1\le j\le M'} X_j' \le B_p) \\ =\ & \int_0^x \bar{F}_M(x-y) dF_M(y) + \bar{F}_M(x)\bar{F}_M(0) \\ & = \int_0^x e^{-\theta^*(x-y)}(1 + u(x-y,p)) dF_M(y) + \bar{F}_M(x)\bar{F}_M(0) \\ \sim\ & \int_0^x e^{-\theta^*(x-y)} dF_M(y) + \bar{F}_M(x)\bar{F}_M(0) \\ & = e^{-\theta^* x}\bar{F}_M(0) - \bar{F}_M(x) + \int_0^x \bar{F}_M(y)\theta^* e^{-\theta^*(x-y)} dy + \bar{F}_M(x)\bar{F}_M(0) \\ =\ & e^{-\theta^* x}\bar{F}_M(0) + \int_0^x \theta^* e^{-\theta^* x}(1 + u(y,p)) dy - \bar{F}_M(x)(1 - \bar{F}_M(0)) \sim e^{-\theta^* x} + \theta^* x e^{-\theta^* x} \end{aligned}$$

uniformly over $x > B_p$. The fourth equality is obtained using integration by parts, and the last equality follows from the observation that $\bar{F}_M(0) = P(\max_{1\le i\le M} X_i' \le B_p) = \sum_{n=0}^{\infty} pq^n F(B_p)^n \to 1$ as $p \to 0$. Noting that $\theta^* \sim p/\mu$, the conclusion of the lemma is then an easy consequence. □

Proof of Lemma (2.19). The inequality is obvious. We will thus focus on the order relation. We first prove that

$$\sum_{n=\lfloor K_p\rfloor+1}^{\infty} pq^n nP(S_{n-1}>x,\max X_i \le \mu+\sqrt{n}) \ll e^{-\theta^* x}+\frac{1}{p}\bar{F}(x)$$

uniformly over $x \ge B_p$. The proof is very similar to that of Lemma (2.5). Adopting the notation there, we can write

$$\begin{aligned}
&\sum_{n=\lfloor K_p\rfloor+1}^{\infty} pq^n nP(S_{n-1}>x,\max X_i \le \mu+\sqrt{n}) \\
=&\sum_{n=\lfloor K_p\rfloor+1}^{\lfloor R_p^2\rfloor\wedge\lfloor l^{-1}(x)\rfloor} pq^n nP(\tilde{S}_{n-1}>x-(n-1)\mu,\max \tilde{X}_i \le \sqrt{n})I(\lfloor R_p^2\rfloor\wedge\lfloor l^{-1}(x)\rfloor>\lfloor K_p\rfloor) \\
&+\sum_{n=\lfloor R_p^2\rfloor+1}^{\lfloor l^{-1}(x)\rfloor} pq^n nP(\tilde{S}_{n-1}>x-(n-1)\mu,\max \tilde{X}_i \le \sqrt{n})I(\lfloor l^{-1}(x)\rfloor>\lfloor R_p^2\rfloor>\lfloor K_p\rfloor) \\
&+\sum_{n=\lfloor K_p\rfloor\vee\lfloor l^{-1}(x)\rfloor+1}^{\infty} pq^n nP(\tilde{S}_{n-1}>x-(n-1)\mu,\max \tilde{X}_i \le \sqrt{n})
\end{aligned}$$

Using the same analysis, the first term will be less than or equal to

$$\begin{cases} \frac{l^{-1}(x)(l^{-1}(x)+1)}{2} pq^{l^{-1}(x)}e^{-\beta\alpha h^2(\sqrt{l^{-1}(x)})} & \text{for } \lfloor l^{-1}(x)\rfloor \le \lfloor R_p^2\rfloor \\ \frac{R_p^2(R_p^2+1)}{2} pe^{(x/\mu)\log q} & \text{for } \lfloor l^{-1}(x)\rfloor > \lfloor R_p^2\rfloor \end{cases}$$

the second term will be less than or equal to

$$\frac{l^{-1}(x)(l^{-1}(x)+1)}{2} pe^{-\beta x r(\sqrt{l^{-1}(x)})}$$

and the third term will be less than or equal to

$$\begin{cases} \left(K_p-\frac{1}{p}+1\right)e^{-pK_p/a'} & \text{for } l^{-1}(x) \ll K_p/a' \\ \left(l^{-1}(x)-\frac{1}{p}+1\right)e^{-pl^{-1}(x)} & \text{for } l^{-1}(x) \ge K_p/a' \end{cases}$$

For the first two terms the same analysis carries over while for the last one we only have to observe again that $px \gg \log x$ for $x \ge K_p/a'$, which will show our claim.

Hence we have

$$\begin{aligned}
&\sum_{n=\lfloor K_p\rfloor+1}^{\infty} pq^n n\int_{B_p}^{x} P(S_{n-1}>x-y,\max X_i \le \mu+\sqrt{n})dF(y) \\
&\ll \int_{B_p}^{x}\left(e^{-\theta^*(x-y)}+\frac{1}{p}\bar{F}(x-y)\right)dF(y) \le \int_{B_p}^{x}\left(\frac{1}{p}+\frac{x-y}{\mu}\right)e^{-\theta^*(x-y)}dF(y)+\frac{1}{p}\bar{F}(x)
\end{aligned}$$

where the last order relation is obtained by using property of class $\mathcal{S}$ that $\int_0^x \bar{F}(x-y)/\bar{F}(x)dF(y) \to 2$ as $x\to\infty$. We conclude our proof of Lemma (2.19). □

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