Integrality gaps for strengthened LP relaxations of Capacitated and Lower-Bounded Facility Location

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Time 3 pm

Location 303 Mudd

Abstract: In *the uncapacitated facility location problem (UFL)* we are given a set of facilities and a set of clients. Every client has to be assigned to some open facility. Opening a facility i incurs a fixed cost f_i, while assigning client j to facility i incurs a cost proportional to their distance c_{ij}. The objective is to minimize the total facility opening and assignment cost.

Metric UFL, where distances satisfy the triangle inequality, enjoys a special stature in approximation algorithms as a testbed of several techniques, among which Linear Programming based methods have been especially prominent. We focus on two generalizations of metric UFL: capacitated facility location (CFL) and lower-bounded facility location (LBFL). In CFL every facility has a capacity which is the maximum number of clients it can serve, while in LBFL every open facility has to serve at least some predetermined number of clients.

Intriguingly, known constant-factor approximation algorithms for CFL and LBFL are all based on local search. The natural LP relaxations for these problems have an unbounded integrality gap. According to Shmoys and Williamson, devising a relaxation-based approximation algorithm for CFL is one of the top 10 open problems in approximation algorithms.

We give the first results on this open problem. We provide substantial evidence against the existence of a good LP relaxation for CFL by showing an unbounded integrality gap for two families of strengthened relaxations. The first family is the hierarchy of LPs resulting from repeated application of the lift-and-project Lovasz-Shrijver procedure, starting from the standard relaxation. We show that the LP relaxation resulting after $Omega(n)$ rounds, where n is the number of facilities in the instance, has an unbounded integrality gap.

We also introduce the family of *proper* LPs which generalize the classic star relaxation, an equivalent form of the natural LP. In the star relaxation

there is a decision variable for every star: a single facility and a set of clients assigned to it. In a proper LP every variable corresponds to what we call a class: an arbitrary set of facilities and clients and an assignment of each client to a facility in the set. The *complexity* of a proper relaxation is the maximum fraction of the available facilities that appear in a class. We show that for CFL and LBFL every proper relaxation of complexity less than 1 has an unbounded integrality gap. On the other hand, there is always a proper LP of complexity 1 that has an integrality gap of 1.

Joint work with: Yannis Moysoglou.