On the Erdös-Lovász Tihany Conjecture for Claw-Free Graphs

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Abstract

In 1968, Erdös and Lovász conjectured that for every graph G and all integers $s, t \geq 2$ such that $s + t - 1 = \chi(G) > \omega(G)$, there exists a partition (S, T) of the vertex set of G such that $\chi(G|S) \geq s$ and $\chi(G|T) \geq t$. For general graphs, the only settled cases of the conjecture are when s and t are small. Recently, the conjecture was proved for a few special classes of graphs: graphs with stability number 2 [1], line graphs [9] and quasi-line graphs [1]. In this paper, we consider the conjecture for claw-free graphs and present some progress on it.

1 Introduction

In 1968, Erdős and Lovász made the following conjecture:

Conjecture 1 (Erdös-Lovász Tihany)**.** *For every graph G with $\chi(G) > \omega(G)$ and any two integers $s, t \geq 2$ with $s + t = \chi(G) + 1$, there is a partition (S, T) of the vertex set $V(G)$ such that $\chi(G|S) \geq s$ and $\chi(G|T) \geq t$.*

Currently, the only settled cases of the conjecture are $(s, t) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$ [2, 10, 11, 12]. Recently, Balogh et. al. proved Conjecture 1 for the class of graphs known as *quasi-line graphs* (a graph is a quasi-line graph if for every vertex v, the set of neighbors of v can be expressed as the union of two cliques). In this paper we consider a class of graphs that is a proper superset of the class of quasi-line graphs: *claw-free graphs*. We prove a weakened version of Conjecture 1 for this class of graphs. Before we state our main result we need to set up some notation and definitions.

In this paper all graphs are finite and simple. Given a graph G, let $V(G)$, $E(G)$ denote the set of vertices and edges of G, respectively. A k-coloring of G is a map $c : V(G) \to \{1, \dots, k\}$ such that for every pair of adjacent vertices $v, w \in V(G)$,

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$c(v) \neq c(w)$. We may also refer to a k-coloring simply as a coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer such that there is a $\chi(G)$-coloring of G.

A *clique* in G is a set of vertices of G that are all pairwise adjacent. A *stable set* in G is a set of vertices that are all pairwise non-adjacent. A set $S \subseteq V(G)$ is an *anti-matching* if every vertex in S is non-adjacent to at most one vertex of S. A *brace* is a clique of size 2, a *triangle* is a clique of size 3 and a *triad* is a stable set of size 3. The *clique number* of G, denoted by $\omega(G)$, is the size of a maximum clique in G, and the *stability number* of G, denoted by $\alpha(G)$ is the size of the maximum stable set in G.

Let G be a graph such that $\chi(G) > \omega(G)$. We say that a brace $\{u, v\}$ is *Tihany* if $\chi(G \backslash \{u, v\}) \geq \chi(G) - 1$. More generally, if K is a clique of size k in G, then we say that K is *Tihany* if $\chi(G \setminus K) \geq \chi(G) - k + 1$.

Let K be a clique in G. We denote by $C(K)$ the set of common neighbors of the members of K, by $A(K)$ the set of their common non-neighbors, and by $M(K)$ the set of vertices that are mixed on the clique K. Formally,

$$\begin{aligned} C(K) =& \{x \in V(G) \backslash K : ux \in E \text{ for all } u \in V(K)\} \\ A(K) =& \{x : ux \notin E \text{ for all } x \in K\} \\ M(K) =& V(G) \setminus (C(K) \cup A(K)). \end{aligned}$$

We say that a clique K is *dense* if $C(K)$ is a clique and we say that it is *good* if $C(K)$ is an anti-matching.

The following theorem is the main result of this paper:

1.1. *Let G be a claw-free graph with $\chi(G) > \omega(G)$. Then there exists a clique K with $|K| \leq 5$ such that $\chi(G \backslash K) > \chi(G) - |K|$.*

To prove 1.1 we use a structure theorem for claw-free graphs (due to the first author and Seymour) that appears in [6] and is described in the next section. Section 3 contains some lemmas that serve as "tools" for later proofs. The next 6 sections are devoted to dealing with the different outcomes of the structure theorem, proving that a (subgraph) minimal counterexample to 1.1 does not fall into any of those classes. Finally, in section 10 all of these results are collected to prove 1.1.

2 Structure Theorem

The goal of this section is to state and describe the structure theorem for claw-free graphs appearing in [6] (or, more precisely, its corollary). We begin with some definitions which are modified from [6].

Let X, Y be two subsets of $V(G)$ with $X \cap Y = \emptyset$. We say that X and Y are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and we say that they are *anticomplete* to each other if no vertex of X is adjacent to a member of Y. Similarly, if $A \subseteq V(G)$ and $v \in V(G) \setminus A$, then v is *complete* to A if v is adjacent to every vertex in A, and *anticomplete* to A if v has no neighbor in A.

Let $F \subseteq V(G)^2$ be a set of unordered pairs of distinct vertices of G such that every vertex appears in at most one pair. Then H is a *thickening* of (G, F) if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$ satisfying the following:

- for each $v \in V(G)$, X_v is a clique of H
- if $u, v \in V(G)$ are adjacent in G and $\{u, v\} \notin F$, then X_u is complete to X_v in H
- if $u, v \in V(G)$ are nonadjacent in G and $\{u, v\} \notin F$, then X_u is anticomplete to X_v in H
- if $\{u, v\} \in F$ then X_u is neither complete nor anticomplete to X_v in H.

Here are some classes of claw-free graphs that come up in the structure theorem.

- **Graphs from the icosahedron.** The *icosahedron* is the unique planar graph with twelve vertices all of degree five. Let it have vertices $v_0, v_1, \ldots, v_{11}$, where for $1 \le i \le 10$, v_i is adjacent to v_{i+1}, v_{i+2} (reading subscripts modulo 10), and v_0 is adjacent to v_1, v_3, v_5, v_7, v_9, and v_{11} is adjacent to $v_2, v_4, v_6, v_8, v_{10}$. Let this graph be G_0. Let G_1 be obtained from G_0 by deleting v_{11} and let G_2 be obtained from G_1 by deleting v_{10}. Furthermore, let $F' = \{\{v_1, v_4\}, \{v_6, v_9\}\}$ and let $F \subseteq F'$.

  Let $G \in \mathcal{T}_1$ if G is a thickening of $(G_0, \emptyset)$, $(G_1, \emptyset)$, or (G_2, F) for some F.

- **Fuzzy long circular interval graphs.** Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \ldots, F_k$ share an endpoint, and no three of them have union Σ. Now let $V \subseteq \Sigma$ be finite, and let H be a graph with vertex set V in which distinct $u, v \in V$ are adjacent precisely if $u, v \in F_i$ for some i.

  Let $F' \subseteq V(H)^2$ be the set of pairs $\{u, v\}$ such that u, v are distinct endpoints of F_i for some i. Let $F \subseteq F'$ such that every vertex of G appears in at most one member of F. Then G is a *fuzzy long circular interval graph* if for some such H and F, G is a thickening of (H, F).

  Let $G \in \mathcal{T}_2$ if G is a fuzzy long circular interval graph.

- **Fuzzy antiprismatic graphs.** A graph K is *antiprismatic* if for every $X \subseteq V(K)$ with $|X| = 4$, X is not a claw and there are at least two pairs of vertices in X that are adjacent. Let H be a graph and let $F \subseteq V(H)^2$ be a set of pairs $\{u, v\}$ such that every vertex of H is in at most one member of F and

  - no triad of H contains u and no triad of H contains v, or
  - there is a triad of H containing both u and v and no other triad of H contains u or v.

Thus F is the set of "changeable edges" discussed in [4]. The pair (H, F) is *antiprismatic* if for every $F' \subseteq F$, the graph obtained from H by changing the adjacency of all the vertex pairs in F' is antiprismatic. We say that a graph G is a *fuzzy antiprismatic graph* if G is a thickening of (H, F) for some antiprismatic pair (H, F).

Let $G \in \mathcal{T}_3$ if G is a fuzzy antiprismatic graph.

Next, we define what it means for a claw-free graph to admit a "strip-structure".

A *hypergraph* H consists of a finite set $V(H)$, a finite set $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (that is, a subset of $V(H) \times E(H)$). For the statement of the structure theorem, we only need hypergraphs such that every member of $E(H)$ is incident with either one or two members of $V(H)$ (thus, these hypergraphs are graphs if we allow "graphs" to have loops and parallel edges). For $F \in E(H)$, $\overline{F}$ denotes the set of all $h \in V(H)$ incident with F.

Let G be a graph. A *strip-structure* (H, η) of G consists of a hypergraph H with $E(H) \neq \emptyset$, and a function η mapping each $F \in E(H)$ to a subset $\eta(F)$ of $V(G)$, and mapping each pair (F, h) with $F \in E(H)$ and $h \in \overline{F}$ to a subset $\eta(F, h)$ of $\eta(F)$, satisfying the following conditions.

(SD1) The sets $\eta(F)$ $(F \in E(H))$ are nonempty and pairwise disjoint and have union $V(G)$.

(SD2) For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \overline{F}$ is a clique of G.

(SD3) For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in G, then there exists $h \in \overline{F_1} \cap \overline{F_2}$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.

There is also a fourth condition, but it is technical and we will not need it in this paper.

Let (H, η) be a strip-structure of a graph G, and let $F \in E(H)$, where $\overline{F} = \{h_1, \ldots, h_k\}$. Let $v_1, \ldots, v_k$ be new vertices, and let J be the graph obtained from $G|\eta(F)$ by adding $v_1, \ldots, v_k$, where v_i is complete to $\eta(F, h_i)$ and anticomplete to all other vertices of J. Then $(J, \{v_1, \ldots, v_k\})$ is called the *strip of* (H, η) *at* F. A strip-structure (H, η) is *nontrivial* if $|E(H)| \geq 2$.

Appendix D contains the descriptions of some strips (J, Z) that we will need for the structure theorem.

We are now ready to state a structure theorem for claw-free graphs that is an easy corollary of the main result of [6].

2.1. *Let G be a connected claw-free graph. Then either*

- *G is a member of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, or*
- *$V(G)$ is the union of three cliques, or*

- *G admits a nontrivial strip-structure such that for each strip (J, Z), $1 \leq |Z| \leq 2$, and if $|Z| = 2$, then either*
  - *$|V(J)| = 3$ and Z is complete to $V(J) \setminus Z$, or*
  - *(J, Z) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$.*

3 Tools

In this section we present a few lemmas that will then be used extensively in the following sections to prove results on the different graphs used in 2.1.

The following result is taken from [12]. Because it is fundamental to many of our results, we include its proof here for completeness.

3.1. *Let G be a graph with chromatic number χ and let K be a clique of size k in G. If K is not Tihany, then every color class of a $(\chi - k)$-coloring of $G \setminus K$ contains a vertex complete to K.*

Proof. Suppose not. Since K is not Tihany, it follows that $G \setminus K$ is $\chi - k$-colorable. Let C be a color class of a $(\chi - k)$-coloring of $G \setminus K$ with no vertex complete to K. Define a coloring of $K \cup C$ by giving a distinct color to each vertex of K and giving each vertex of C a color of one of its non-neighbors in K. This defines a k-coloring of $G|(K \cup C)$. Note also that $G \setminus (K \cup C)$ is $\chi - k - 1$-colorable. However, this implies that G is $(\chi - 1)$-colorable, a contradiction. This proves 3.1. □

The next lemma is one of our most important and basic tool.

3.2. *Let G be a graph such that $\chi(G) > \omega(G)$. Let K be a clique of G. If K is dense, then it is Tihany.*

Proof. Suppose that K is not Tihany. Let $\mathcal{C}$ be a $\chi - k$-coloring of $G \setminus K$. By 3.1, every color class of $\mathcal{C}$ contains a vertex complete to K. Hence, every color class contains a member of $C(K)$ and so $|C(K) \cup K| \geq \chi(G) > \omega(G)$, a contradiction. This proves 3.2. □

Let (A, B) be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if A, B are cliques, and for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete. A *W-join* (A, B) is a homogeneous pair in which A is neither complete nor anticomplete to B. We say that a W-join (A, B) is *reduced* if we can partition A into two sets A_1 and A_2 and we can partition B into B_1, B_2 such that A_1 is complete to B_1, A_2 is anticomplete to B, and B_2 is anticomplete to A. Note that since A is neither complete nor anticomplete to B, it follows that both A_1 and B_1 are non-empty and at least one of A_2, B_2 is non-empty. We call a W-join that is not reduced a *non-reduced* W-join.

Let H be a thickening of (G, F) and let $\{u, v\} \in F$. Then we notice that (X_u, X_v) is a W-join in H. If for every $\{u, v\} \in F$ we have that (X_u, X_v) is a reduced W-join then we say that H is a *reduced thickening* of G.

The following result appears in [3].

3.3. *Let G be a claw-free graph and suppose that G admits a non-reduced W-join. Then there exists a subgraph H of G with the following properties:*

1. *H is a claw-free graph, $|V(H)| = |V(G)|$ and $|E(H)| < |E(G)|$.*
2. *$\chi(H) = \chi(G)$.*

The result of 3.3 implies the following:

3.4. *Let G be a claw-free graph with $\chi(G) > \omega(G)$ that is a minimal counterexample to 1.1. Assume also that G is a thickening of (H, F) for some claw-free graph H and $F \subseteq V(H)^2$. Then G is a reduced thickening of (H, F).*

For a clique K and $F \subseteq V(G)^2$, we define $S_F(K) = \{x : \exists k \in K \text{ s.t. } \{x, k\} \in F \text{ and } x \in C(K \backslash k)\}$.

3.5. *Let G be a reduced thickening of (H, F) for some claw-free graph H and $F \subseteq V(H)^2$. Let K be a clique in H such that for all $x, y \in C(K)$, $\{x, y\} \notin F$. If $C(K) \cup S_F(K)$ is a clique, then there exists a dense clique of size $|K|$ in G.*

Proof. Let K' be a clique of size $|K|$ in G such that $K' \cap X_v \neq \emptyset$ for all $v \in K$. By the definition of a thickening such a clique exists. Moreover since $C(K) \cup S_F(K)$ is a clique, it follows that K' is dense. This proves 3.5. □

The following lemma is a direct corollary of 3.2 and 3.5.

3.6. *Let G be a reduced thickening of (H, F) for some claw-free graph H and $F \subseteq V(H)^2$. Let K be a dense clique in H such that for all $x, y \in C(K)$, $\{x, y\} \notin F$. If $C(K) \cup S_F(K)$ is a clique, then there exists a Tihany clique of size $|K|$ in G.*

The following result helps us handle the case when $C(x)$ is an antimatching for some vertex $x \in V(G)$.

3.7. *Let G be a graph with $\chi(G) > \omega(G)$. Let $u, x, y \in V(G)$ such that $ux, uy \in E(G)$ and $xy \notin E(G)$. Let $E = \{u, x\}$ and $E' = \{u, y\}$. If $C(E) = C(E')$ then E, E' are Tihany.*

Proof. Suppose that E is not Tihany. Let $\mathcal{C}$ be a $(\chi(G)-2)$-coloring of $G \setminus \{u, x\}$. Let $C \in \mathcal{C}$ be the color class such that $y \in C$. By Lemma 3.1, there is a vertex $z \in C$ such that z is complete to E, and so $z \in C(E)$. But y is complete to $C(E)$, a contradiction. Hence E is be Tihany and by symmetry, so is E'. □

In particular, if we have a vertex x such that $C(x)$ is an antimatching, we can find a Tihany edge either by 3.2 or by 3.7.

3.8. *Let H be a graph, G a thickening of (G, F) for some valid $F \subseteq G(V)^2$ such that $\chi(G) > \omega(G)$. Let K be a clique of H. Assume that for all $\{x, y\} \in F$ such that $x \in K$, y is complete to $C(K) \backslash \{y\}$. Let $u, v \in C(K)$ such that u is not adjacent to v and $\{u, v\}$ is complete to $C(K) \backslash \{u, v\}$. Moreover assume that if there exists $E \in F$ with $\{u, v\} \cap E \neq \emptyset$, then $E = \{u, v\}$. Then there exists a Tihany clique of size $|K|+1$ in G.*

Proof. Assume not. Let K' be a clique of size K in G such that $K' \cap X_y \neq \emptyset$ for all $y \in K$. If $\{u,v\} \notin F$, let $a \in X_u$, $A = X_u$, $b \in X_v$ and $B = X_v$. If $\{u,v\} \in F$, let X_u^1, X_u^2, X_v^1 and X_v^2 be as in the definition of reduced W-join. By symmetry, we may assume that X_u^2 is not empty. If X_v^2 is empty, let $a \in X_u^2$, $A = X_u^2$, $b \in X_v^1$ and $B = X_v^1$; and if X_v^2 is not empty, let $a \in X_u^2$, $A = X_u$, $b \in X_v^2$ and $B = X_v$.

Now let $T_a = K' \cup \{a\}$ and $T_b = K' \cup \{b\}$. We may assume that $\chi(G\backslash T_a) = \chi(G\backslash T_b) = \chi(G) - |K| - 1$. By 3.1, we may assume that every color class $G\backslash T_a$ contains a common neighbor of T_a. Since no vertex of B is complete to T_1, and since B is a clique complete to $C(T_1)\backslash A$, it follows that $|A| > |B|$. But similarly, $|B| > |A|$, a contradiction. This proves 3.8. □

We need an additional definition before proving the next lemma. Let K be a clique; we denote by $\overline{C}(K)$ the closed neighborhood of K, i.e. $\overline{C}(K) := C(K) \cup K$.

3.9. *Let G be a graph such that $\chi(G) > \omega(G)$. Let A and B be cliques such that $2 \leq |A|, |B| \leq 3$ (i.e., each one is a brace or a triangle). If $\overline{C}(A) \cap \overline{C}(B) = \emptyset$ and $\overline{C}(A) \cup \overline{C}(B)$ contains no triads then at least one of A, B is Tihany.*

Proof. Assume not and let $k = \chi(G) - |A|$. By 3.1, in every k-coloring of $G\backslash A$ every color class must have a vertex in $C(A)$. As there is no triad in $\overline{C}(A) \cup \overline{C}(B)$, it follows that every vertex of $C(A)$ is in a color class with at most one vertex of $\overline{C}(B)$, thus $\overline{C}(A) > \overline{C}(B)$. By symmetry, it follows that $\overline{C}(A) < \overline{C}(B)$, a contradiction. This proves 3.9. □

3.10. *Let G be a claw-free graph such that $\chi(G) > \omega(G)$. If G admits a clique cutset, then there is a Tihany brace in G.*

Proof. Let K be a clique cutset. Let $A, B \subset V(G)\backslash K$ such that $A \cap B = \emptyset$ and $A \cup B \cup K = V(G)$. Let $\chi_A = \chi(G|(A \cup K))$ and $\chi_B = \chi(G|(B \cup K))$. By symmetry, we may assume that $\chi_A \geq \chi_B$.

(1) $\chi(G) = \chi_A$

Let $\mathcal{S}_A = (A_1, A_2, \ldots, A_{\chi_A})$ and $\mathcal{S}_B = (B_1, B_2, \ldots, B_{\chi_B})$ be optimal coloring of $G|(A \cup K)$ and $G|(B \cup K)$. Let $K = \{k_1, k_2, \ldots, K_{|K|}\}$. Up to renaming the stable sets, we may assume that $A_i \cap B_i = \{k_i\}$ for all $i = 1, 2, \ldots, |K|$. Then $\mathcal{S} = (A_1 \cup B_1, A_2 \cup B_2, \ldots, A_{\chi_B} \cup B_{\chi_B}, A_{\chi_B+1}, \ldots, A_{\chi_A}\}$ is a χ_A-coloring of G. This proves (1).

Now let $x \in B$ and $y \in K$ be such that $xy \in E(G)$. Then $\chi(G\backslash\{x,y\}) \geq \chi(G|(A \cup K\backslash\{y\}) \geq \chi_A - 1 \geq \chi(G) - 1$. Hence $\{x,y\}$ is a Tihany brace. This proves 3.10. □

4 The Icosahedron and Long Circular Interval Graphs

4.1. *Let $G \in \mathcal{T}_1$. If $\chi(G) > \omega(G)$, then there exists a Tihany brace in G.*

Proof. Let $v_0, v_1, \dots, v_{11}$ be as in the definition of the icosahedron. Let G_0, G_1, G_2, and F be as in the definition of $\mathcal{T}_1$. Then G is a thickening of either $(G_0, \emptyset)$, $(G_1, \emptyset)$, or (G_2, F) for $F \subseteq \{(v_1, v_4), (v_6, v_9)\}$. For $0 \leq i \leq 11$, let X_{v_i} be as in the definition of thickening (where $X_{v_{11}}$ is empty when G is a thickening of $(G_1, \emptyset)$ or (G_2, F), and $X_{v_{10}}$ is empty when G is a thickening of (G_2, F)). Let $x_i \in X_{v_i}$ and $w_i = |X_{v_i}|$.

First suppose that G is a thickening of $(G_1, \emptyset)$ or (G_2, F). Then $C(\{x_4, x_6\}) = X_{v_4} \cup X_{v_5} \cup X_{v_6}$ is a clique. Therefore, $\{x_4, x_6\}$ is a Tihany brace by 3.2.

So we may assume that G is a thickening of $(G_0, \emptyset)$. Suppose that no brace of G is Tihany and let $E = \{x_1, x_3\}$. Then $G \backslash E$ is $(\chi - 2)$-colorable. By 3.1, every color class contains at least one vertex from $C(E) = (X_1 \cup X_2 \cup X_3 \cup X_0) \setminus \{x_1, x_3\}$. Since $\alpha(G) = 3$, it follows that every color class has at most two vertices from $\bigcup_{i=4}^{11} X_{v_i}$. Hence we conclude that

$$w_4 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10} + w_{11} \leq 2 \cdot (w_1 + w_2 + w_3 + w_0 - 2)$$

A similar inequality exists for every brace $\{x_i, x_j\}$. Summing these inequalities over all braces $\{x_i, x_j\}$, it follows that $(\sum_{i=0}^{11} 20w_i) \leq (\sum_{i=0}^{11} 20w_i) - 120$, a contradiction. This proves 4.1. $\square$

4.2. *Let $G \in \mathcal{T}_2$. If $\chi(G) > \omega(G)$, then there exists a Tihany brace in G.*

Proof. Let $H, F, \Sigma, F_1, \dots, F_k$ be as in the definition of $\mathcal{T}_2$ such that G is a thickening of (H, F). Let F_i be such that there exists no j with $F_i \subset F_j$. Let $\{x_k, \dots, x_l\} = V(H) \cap F_i$ and without loss of generality, we may assume that $\{x_k, \dots, x_l\}$ are in order on Σ. Since $C(\{x_k, x_l\}) = \{x_{k+1}, \dots, x_{l-1}\}$, it follows that $\{x_k, x_l\}$ is dense. Hence by 3.6 there exists a Tihany brace in G. This proves 4.2. $\square$

5 Non-2-substantial and Non-3-substantial Graphs

In this section we study graphs where a few vertices cover all the triads. An antiprismatic graph G is *k-substantial* if for every $S \subseteq V(G)$ with $|S| < k$ there is a triad T with $S \cap T = \emptyset$. The *matching number* of a graph G, denoted by $\mu(G)$, is the number of edges in a maximum matching in G. Balogh et al. [1] proved the following theorem.

5.1. *Let G be a graph such that $\alpha(G) = 2$ and $\chi(G) > \omega(G)$. For any two integers $s, t \geq 2$ such that $s + t = \chi(G) + 1$ there exists a partition (S, T) of $V(G)$ such that $\chi(G|S) \geq s$ and $\chi(G|T) \geq t$.*

The following theorem is a result of Gallai and Edmonds on matchings and it will be used in the study of non-2-substantial and non-3-substantial graphs.

5.2 (Gallai-Edmonds Structure Theorem [7], [8])**.** *Let $G = (V, E)$ be a graph. Let D denote the set of nodes which are not covered by at least one maximum matching of G. Let A be the set of nodes in $V \backslash D$ adjacent to at least one node in D. Let $C = V \backslash (A \cup D)$. Then:*

- *i) The number of covered nodes by a maximum matching in G equals to $|V| + |A| - c(D)$, where $c(D)$ denotes the number of components of the graph spanned by D.*

ii) If M is a maximum matching of G, then for every component F of D, $E(D) \cap M$ covers all but one of the nodes of F, $E(C) \cap M$ is a perfect matching and M matches all the nodes of A with nodes in distinct components of D.

5.3. *Let G be an antiprismatic graph. Let K be a clique and $u, v \in V(G) \backslash \overline{C}(K)$ be non-adjacent. If $\alpha(G|(C(K) \cup \{u, v\})) = 2$ and $\alpha(G|K \cup \{u, v\}) = 3$, then $G|\overline{C}(K)$ is cobipartite.*

Proof. Since there is no triad in $C(K) \cup \{u, v\}$, we deduce that there is no vertex in $C(K)$ anticomplete to $\{u, v\}$. Since G is claw-free and $\alpha(G|K \cup \{u, v\}) = 3$, it follows that there is no vertex in $C(K)$ complete to $\{u, v\}$. Let $C_u, C_v \subseteq C(K)$ be such that $C_u \cup C_v = C(K)$ and for all $x \in C(K)$, x is adjacent to u and non-adjacent to v if $x \in C_u$, and x is adjacent to v and non-adjacent to u if $x \in C_v$. Since $\alpha(G|(C_v \cup \{u\})) = 2$, we deduce that C_v is a clique and by symmetry C_u is a clique. Hence $\overline{C}(K)$ is the union of two cliques. This proves 5.3. □

5.4. *Let G be a claw-free graph such that $\chi(G) > \omega(G)$. Let K be a clique such that $\alpha(G \backslash K) \leq 2$. Then there exists a Tihany clique of size at most $|K| + 1$ in G.*

Proof. Assume not. Let $n = |V(G)|$, $w \in C(K)$ and $K' = K \cup \{w\}$ (such a vertex w exists since K is not Tihany).

(1) $\chi(G) = n - \mu(G^c)$.

Since K' is not Tihany, it follows that $\chi(G \backslash K') = \chi(G) - |K'|$. Since $\alpha(G \backslash K') \leq 2$, we deduce that $\chi(G \backslash K') \geq \frac{n-|K'|}{2}$, and thus $\chi(G) \geq \frac{n+|K'|}{2}$. Hence in every optimal coloring of G the color classes have an average size strictly smaller than 2, and since G is claw-free, we deduce that there is an optimal coloring of G where all color classes have size 1 or 2. It follows that $\chi(G) \leq n - \mu(G^c)$. But clearly $\chi(G) \geq n - \mu(G^c)$, thus $\chi(G) = n - \mu(G^c)$. This proves (1).

(2) *Let T be a clique of size $|K| + 1$ in G, then $\chi(G \backslash T) = n - |T| - \mu(G^c \backslash T)$.*

Since T is not Tihany, it follows that $\chi(G \backslash T) = \chi(G) - |T| \geq \frac{n+|K'|}{2} - |T| = \frac{n-|T|}{2} = \frac{|V(G \backslash T)|}{2}$. Hence in every optimal coloring of $G \backslash T$, the color classes have an average size smaller than 2, and since G is claw-free, we deduce that there is an optimal coloring of $G \backslash T$ where all color classes have size 1 or 2. It follows that $\chi(G \backslash T) \leq |V(G \backslash T| - \mu(G^c \backslash T)$. Hence $\chi(G \backslash T) = n - |T| - \mu(G^c \backslash T)$. This proves (2).

Let A, D, C be as in 5.2. Since $\chi(G) \geq \frac{n+|K'|}{2}$ and $\chi(G) = n - \mu(G^c)$, we deduce that $\mu(G^c) \leq \frac{n-|K'|}{2}$. By 5.2 i), we deduce that $\mu(G^c) = \frac{n+|A|-c(D)}{2}$. Thus, it follows that $c(D) \geq |K'|$. Let $D_1, D_2, \dots, D_{c(D)}$ be the anticomponents of D. Let $d_i \in D_i$ for $i = 1, \dots, c(D)$.

(3) $|D_i| = 1$ *for all i.*

Assume not and by symmetry assume that $|D_1| > 1$. Since G is claw-free, we deduce that $\alpha(G|D_1) = 2$. Thus there exist $x, y \in D_1$ such that x is adjacent to y. Now $T = \{x, y, d_2, \ldots, d_{|K|}\}$ is a clique of size $|K| + 1$ and by 5.2 ii), it follows that $\mu(G^c \backslash T) < \mu(G^c)$. By (1) and (2), it follows that $\chi(G\backslash T) + |T| = n - \mu(G^c\backslash T) > n - \mu(G^c) = \chi(G)$, a contradiction. This proves (3).

Let $T = \{d_1, \ldots, d_{|K|+1}\}$. By (3), it follows that $C(T) \cap D$ is a clique. By 3.2, we deduce that $C(T) \cap A \neq \emptyset$. Let $x \in C(T) \cap A$. Now $S = \{d_1, \ldots, d_{|K|}, x\}$ is a clique of size $|K| + 1$ and by 5.2 ii), it follows that $\mu(G^c\backslash S) < \mu(G^c)$. By (1) and (2), it follows that $\chi(G\backslash S) + |S| = n - \mu(G^c\backslash S) > n - \mu(G^c) = \chi(G)$, a contradiction. This concludes the proof of 5.4. □

5.5. *Let H be a claw-free graph such that there exists $x \in V(H)$ with $\alpha(H\backslash x) = 2$. Let G be a reduced thickening of H such that $\chi(G) > \omega(G)$ and $|X_x| > 1$. Then for all $\{u, v\} \in X_x$, $\chi(G\backslash\{u, v\}) \geq \chi(G) - 1$.*

Proof. Let $u, v \in X_x$. We may assume that $\{u, v\}$ is not Tihany. Let $k = \chi(G\backslash\{u, v\})$ and $\mathcal{S} = (S_1, S_2, \ldots, S_k)$ be a k-coloring of $G\backslash\{u, v\}$. By 3.1, $S_i \cap C(\{u, v\}) \neq \emptyset$. Let $I_l = \{i : |S_i| = l\}$ and let $O = C(\{u, v\}) \cap \bigcup_{i \in I_1 \cup I_2} S_i$ and $P = C(\{u, v\}) \cap \bigcup_{i \in I_3} S_i$.

Since $\alpha(H\backslash x) = 2$, it follows that $S_i \cap X_x \neq \emptyset$ for all $i \in I_3$. Hence, P is a clique complete to O and thus $\omega(G|O \cup P) = \omega(G|O) + |I_3|$. Since $\chi(G) > \omega(G)$, we deduce that $\omega(G|O) < |I_1 \cup I_2|$. By 5.3 and since $O \subseteq \overline{C}(X_x)$, we deduce that $G|O$ is cobipartite. Hence $\chi(G|O) = \omega(G|O) < |I_1 \cup I_2|$. Thus the coloring $\mathcal{S}$ does not induce an optimal coloring of $G|O$. It follows that there exists an augmenting antipath $P = p_1 - p_2 - \ldots - p_{2l}$ in O. Now let $T_i = \{p_{2i-1}, p_{2i}\}$ for $i = 1, \ldots, l$. Let s be such that $p_1 \in S_s$ and e be such that $p_{2l} \in S_e$. They are the color classes where the augmenting antipath starts and ends. If $|S_s| = 2$, let $T_{l+1} = (\{u\} \cup S_s\backslash p_1)$, otherwise let $T_{l+1} = \{u\}$. If $|S_e| = 2$, let $T_{l+2} = (\{v\} \cup S_e\backslash p_{2l})$, otherwise let $T_{l+2} = \{v\}$. Let $J = \{i | S_i \cap V(P) \neq \emptyset\}$. Clearly $|J| = l + 1$. Now $(T_1, T_2, \ldots, T_{l+2})$ is a (l+2)-coloring of $\bigcup_{i \in J} S_i \cup \{u, v\}$, which together with the color classes S_i for $i \notin J$ create a $k + 1$-coloring of G, a contradiction. This proves 5.5. □

The next lemma is a direct corollary of 5.4 and 5.5.

5.6. *Let H be a non-2-substantial claw-free graph. Let G be a reduced thickening of an augmentation of H such that $\chi(G) > \omega(G)$. Then there exists a Tihany brace in G.*

Now we look at non-3-substantial graphs.

5.7. *Let H be a non-3-substantial antiprismatic graph. Let $u, v \in H$ be such that $\alpha(H\backslash\{u, v\}) = 2$. Let G be a reduced thickening of H such that $\chi(G) > \omega(G)$. If u is not adjacent to v, then there exists a Tihany brace or triangle in G.*

Proof. Assume not. Let $N_u = C(u)\backslash C(\{u, v\}$ and $N_v = C(u)\backslash C(\{u, v\})$. Since H is antiprismatic, it follows that N_u and N_v are antimatchings.

By 5.6, we deduce that N_u and N_v are not cliques. Let $x_u, y_u \in N_u$ be not adjacent, and $x_v, y_v \in N_v$ be not adjacent. Since $\alpha(H\backslash\{u,v\}) = 2$ and H is antiprismatic, we may assume by symmetry that $x_u x_v, y_u y_v$ are edges, and $x_u y_v, y_u x_v$ are non-edges. Since $\alpha(h\backslash\{u,v\}) = 2$ and H is antiprismatic, it follows that every vertex in $C(\{u,v\})$ is either strongly complete to $x_u x_v$ and strongly anticomplete to $y_u y_v$, or strongly complete to $y_u y_v$ and strongly anticomplete to $x_u x_v$. Let (N_x, N_y) be the partition of $C(\{u,v\})$ such that all $x \in N_x$ are complete to $x_u x_v$ and and all $y \in N_y$ are complete to $y_u y_v$.

Assume first that $N_x \neq \emptyset$ and $N_y \neq \emptyset$. Let $n_x \in N_x$ and $n_y \in N_y$ and let $T_u = \{u, y_u, n_y\}$ and $T_v = \{v, x_v, n_x\}$. Clearly T_u and T_v are triangles.

(1) *$\alpha(G|(\overline{C}(T_u) \cup \overline{C}(T_v)) = 2$ and $\overline{C}(T_u) \cap \overline{C}(T_v) = \emptyset$.*

Assume not. Since $\overline{C}(T_u) \subseteq N_y \cup N_u \cup \{u\}$ and $\overline{C}(T_v) \subseteq N_x \cup N_v \cup \{v\}$, we deduce that $\overline{C}(T_u) \cap \overline{C}(T_v) = \emptyset$. Let $T \in \overline{C}(T_u) \cup \overline{C}(T_v)$ be a triad. By symmetry, we may assume that $u \in T$. Clearly, $T\backslash u \in N_v$. But since H is antiprismatic, we deduce that $T\backslash u \subseteq C(n_x)$, hence $T\backslash u \notin \overline{C}(T_u) \cup \overline{C}(T_v)$, a contradiction. This proves (1).

Now let $S_u, S_v \in G$ be triangles such that $|S_u \cap X_u| = |S_u \cap X_{y_u}| = |S_u \cap X_{n_y}| = 1$ and $|S_v \cap X_v| = |S_v \cap X_{x_v}| = |S_v \cap X_{n_x}| = 1$. By (1) and 3.9 and since G is a reduced thickening of H, we deduce that there is a Tihany triangle in G.

Now assume that at least one of N_x, N_y is empty. By symmetry, we may assume that N_x is empty. Since $C(\{u, x_u\})$ is an antimatching, by 3.8 there exists a Tihany triangle in G. This concludes the proof of 5.7. □

5.8. *Let H be a non-3-substantial antiprismatic graph. Let $u, v \in H$ be such that $\alpha(G\backslash\{u,v\}) = 2$. Let G be a reduced thickening of (H, F) for some valid $F \subseteq V(G)^2$ such that $\chi(G) > \omega(G)$. If u is adjacent to v, then there exists a Tihany clique K in G with $|K| \leq 4$.*

Proof. Assume not. By 5.4, we may assume that $|X_u \cup X_v| > 2$. By 5.6, we may assume that $|X_u| > 0$ or $|X_v| > 0$. If $|X_u| = 1$, then $G\backslash X_u$ is a reduced thickening of a non-2-substantial antiprismatic graph. By 5.5, there exists a brace $\{x, y\}$ in X_v such that $\chi(G\backslash(\{x,y\} \cup X_u)) \geq \chi(G\backslash X_u) - 1$. But $\chi(G\backslash X_u) - 1 \geq \chi(G) - 2$, hence $\{x, y\} \cup X_u$ is a Tihany triangle, a contradiction. Thus $|X_u| > 1$, and by symmetry $|X_v| > 1$.

Let $x_1, y_1 \in X_u$ and $x_2, y_2 \in X_v$, thus $C = \{x_1, x_2, y_1, y_2\}$ is a clique of size 4.

Let $k = \chi(G\backslash C)$ and $\mathcal{S} = (S_1, S_2, \dots, S_k)$ be a k-coloring of $G\backslash C$. By 3.1, $S_i \cap N(C) \neq \emptyset$. For $l = 1, 2, 3$ let $I_l = \{i : |S_i| = l\}$ and let $O = N(C) \cap \bigcup_{i \in I_1 \cup I_2} S_i$ and $P = N(C) \cap \bigcup_{i \in I_3} S_i$.

Since $\alpha(H\backslash\{u,v\}) = 2$, it follows that $S_i \cap (X_u \cup X_v) \neq \emptyset$ for all $i \in I_3$. Hence, $\omega(G|O \cup P) = \omega(G|O) + |I_3|$. Since $\chi(G) > \omega(G)$, we deduce that $\omega(G|O) < |I_1 \cup I_2|$. By 5.3, we deduce that $G|O$ is cobipartite. Hence $\chi(G|O) = \omega(G|O) < |I_1| + |I_2|$. Thus the coloring $\mathcal{S}$ does not induce an optimal coloring of $G|O$. It follows that there exists an augmenting antipath $P = p_1 - p_2 - \dots - p_{2l}$ in O. Now let $T_i = \{p_{2i-1}, p_{2i}\}$ for $i = 1, \dots, l$. Let s be such that $p_1 \in S_s$ and e be such that $p_{2l} \in S_e$. They are the color classes where the augmenting antipath starts and ends. Since $S_s\backslash p_1$ is not complete to $\{x_1, y_1\}$, we deduce that there exists $\hat{s} \in \{1, 2\}$ such that $x_{\hat{s}}$ is antiadjacent to $S_s\backslash p_1$. Let $T_{l+1} = \{x_{\hat{s}}\} \cup S_s\backslash p_1$ and $T_{l+2} = \{x_1, x_2\}\backslash x_{\hat{s}}$. Since $S_e\backslash p_{2l}$ is not complete to $\{x_2, y_2\}$, we deduce that there exists $\hat{e} \in \{1, 2\}$ such that $x_{\hat{e}}$ is antiadjacent to $S_e\backslash p_{2l}$. Let $T_{l+3} = \{x_{\hat{e}}\} \cup S_e\backslash p_{2l}$ and $T_{l+4} = \{y_1, y_2\}\backslash x_{\hat{e}}$.

Let $J = \{i|S_i \cap V(P) \neq \emptyset\}$. Clearly $|J| = l + 1$. Now $(T_1, T_2, \dots, T_{l+2}, T_{l+3}, T_{l+4})$ is a (l+4)-coloring of $\bigcup_{i\in J} S_i \cup \{x_1, x_2, y_1, y_2\}$, which together with the color classes S_i, for $i \notin J$, create a $k + 3$-coloring of G, a contradiction. This proves 5.8. □

The following lemma is a direct corollary of 5.7 and 5.8.

5.9. *Let H be a non-3-substantial antiprismatic graph. Let G be a reduced thickening of H such that $\chi(G) > \omega(G)$. Then there exists a Tihany clique $K \subset V(G)$ with $|K| \leq 4$.*

6 Complements of orientable prismatic graphs

In this section we study the complements of orientable prismatic graphs. A graph is *prismatic* if its complement is antiprismatic. Let G be a graph. The *core* of G is the union of all the triangles in G. If $\{a, b, c\}$ is a triangle in G and both b, c only belong to one triangle in G, then b and c are said to be *weak*. The *strong core* of G is the subset of the core such that no vertex in the strong core is weak. As proved in [4], if H is a thickening of (G, F) for some valid $F \subseteq V(G)^2$ and $\{x, y\} \in F$, then x and y are not in the strong core.

A *three-cliqued claw-free graph* (G, A, B, C) consists of a claw-free graph G and three cliques A, B, C of G, pairwise disjoint and with union $V(G)$. The complement of a tree-cliqued graph is a *3-coloured graph*. Let $n \geq 0$, and for $1 \leq i \leq n$, let (G_i, A_i, B_i, C_i) be a three-cliqued graph, where $V(G_1), \dots, V(G_n)$ are all nonempty and pairwise vertex-disjoint. Let $A = A_1 \cup \dots \cup A_n$, $B = B_1 \cup \dots \cup B_n$, and $C = C_1 \cup \dots \cup C_n$, and let G be the graph with vertex set $V(G_1) \cup \dots \cup V(G_n)$ and with adjacency as follows:

- for $1 \leq i \leq n$, $G|V(G_i) = G_i$;
- for $1 \leq i < j \leq n$, A_i is complete to $V(G_j) \setminus B_j$; B_i is complete to $V(G_j) \setminus C_j$; and C_i is complete to $V(G_j) \setminus A_j$; and
- for $1 \leq i < j \leq n$, if $u \in A_i$ and $v \in B_j$ are adjacent then u, v are both in no triads; and the same applies if $u \in B_i$ and $v \in C_j$, and if $u \in C_i$ and $v \in A_j$.

In particular, A, B, C are cliques, and so (G, A, B, C) is a three-cliqued graph and (G^c, A, B, C) is a 3-coloured graph; we call the sequence (G_i, A_i, B_i, C_i) $(i = 1, \dots, n)$ a *worn hex-chain* for (G, A, B, C). When $n = 2$ we say that (G, A, B, C) is a *worn hex-join* of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). Similarly, the sequence (G_i^c, A_i, B_i, C_i) $(i = 1, \dots, n)$ is a *worn hex-chain* for (G^c, A, B, C), and when $n = 2$, (G^c, A, B, C) is a *worn hex-join* of (G_1^c, A_1, B_1, C_1) and (G_2^c, A_2, B_2, C_2). Note also that every triad of G is a triad of one of $G_1, \dots, G_n$. If each G_i is claw-free then so is G and if each G_i^c is prismatic then so is G^c.

If (G, A, B, C) is a three-cliqued graph, and $\{A', B', C'\} = \{A, B, C\}$, then (G, A', B', C') is also a three-cliqued graph, that we say is a *permutation* of (G, A, B, C).

A list of the definitions needed for the study of the prismatic graphs can be found in appendix A. The structure of prismatic graph has been extensively studied in [4] and [5]; the resulting two main theorems are the following.

6.1. *Every orientable prismatic graph that is not 3-colourable is either not 3-substantial, or a cycle of triangles graph, or a ring of five graph, or a mantled $L(K_{3,3})$.*

6.2. *Every 3-coloured prismatic graph admits a worn chain decomposition with all terms in $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$.*

In the remainder of the section, we use these two results to prove our main theorem for complements of orientable prismatic graphs. We begins with some results that deal with the various outcomes of 6.1.

6.3. *Let H be a prismatic cycle of triangles and G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \in V(G)^2$ such that $\chi(G) > \omega(G)$. Then there exists a Tihany brace or triangle in G.*

Proof. Let the set X_i be as in the definition of a cycle of triangles. Up to renaming the sets, we may assume $|\hat{X}_{2n}| = |\hat{X}_4| = 1$. Let $u \in \hat{X}_{2i}$ and $v \in \hat{X}_4$; hence uv is an edge. We have

$$C_H(\{u, v\}) = \bigcup_{j=1 \mod 3, j \geq 4} X_j \cup R_1 \cup L_3.$$

If $|\hat{X}_2| > 1$, then $|R_1| = |L_3| = \emptyset$ and so $C_H(\{u, v\})$ is a clique. Therefore by 3.6, there is a Tihany brace in G. If $|\hat{X}_2| = 1$, the only non-edges in $\overline{G}|C_H(\{u, v\})$ are a perfect anti-matching between R_1 and L_3. Hence by 3.8, there is a Tihany triangle in G. This proves 6.3 □

6.4. *Let H be a ring of five graph. Let G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \in V(G)^2$ such that $\chi(G) > \omega(G)$. Then there is a Tihany triangle in G.*

Proof. Let a_2, b_3, a_4 be as in the definition of a ring of five. $C(\{a_2, b_3, a_4\}) = V_2 \cup V_4$ and thus $\{a_2, b_3, a_4\}$ is a dense triangle. By the definitions of H and F, it follows that $\{a_2, b_3, a_4\} \cap E = \emptyset$ for all $E \in F$. Hence by 3.6, there exists a Tihany triangle in G. This proves 6.4. □

6.5. *Let H be a mantled $L(K_{3,3})$ and G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \in V(G)^2$. If $\chi(G) > \omega(G)$, then there exists a Tihany brace in G.*

Proof. Let W, a_j^i, V^i, V_i be as in the definition of mantled $L(K_{3,3})$. Let X_j^i be the clique corresponding to a_j^i in the thickening and $\mathcal{W}$ (resp. $\mathcal{V}_i$, $\mathcal{V}^i$) be the set of vertices corresponding to W (resp. V_i, V^i) in the thickening. Let $x_i^j \in X_i^j$, $\mathcal{V} = \cup_{i=1}^3 \mathcal{V}_i \cup \mathcal{V}^i$ and $k = \chi(G)$.

For a brace E in G, let $M_W(E) := M(E) \cap \mathcal{W}$, $M_V(E) := M(E) \cap \mathcal{V}$, $A_W(E) := A(E) \cap \mathcal{W}$ and $A_V(E) := A(E) \cap \mathcal{V}$. Let $E = \{x_i^j, x_{i'}^{j'}\}$ and let S be a color class in a (k-2)-colouring of $G \backslash E$.

(1) *If $S \cap A_V(E) \neq \emptyset$, then $|S| \leq 2$.*

Assume not. Let $S = \{x, y, z\}$ and without loss of generality we may assume that $E = \{x_1^1, x_2^1\}$ and $x \in A_V(E) = \mathcal{V}^1$. Since x is complete to $\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ and X_i^j, for $i = 1, 2, 3$ $j = 2, 3$, we deduce that $y, z \notin \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ and $y, z \notin X_i^j$, for $i = 1, 2, 3$ $j = 2, 3$. Since there is no triad in $\mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3$, it follows that $|\{y, z\} \cap (\mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3)| \leq 1$. Since $X_1^1 \cup X_2^1 \cup X_3^1$ is a clique, we deduce that $|\{y, z\} \cap (X_1^1 \cup X_2^1 \cup X_3^1)| \leq 1$. Hence, we may assume by symmetry that $y \in X_1^1 \cup X_2^1 \cup X_3^1$ and $z \in \mathcal{V}^2 \cup \mathcal{V}^3$. But $X_1^1 \cup X_2^1 \cup X_3^1$ is complete to $\mathcal{V}^2 \cup \mathcal{V}^3$, a contradiction. This proves (1).

(2) *If $S \cap M_V(E) \neq \emptyset$, then $|S| \leq 2$.*

Assume not. Let $S = \{x, y, z\}$ and without loss of generality we may assume that $E = \{x_1^1, x_2^1\}$ and $x \in \mathcal{V}_1$. Since x is complete to $\mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3$ and $X_2^j \cup X_3^j$, for $j = 1, 2, 3$, we deduce that $y, z \notin \mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3$ and $y, z \notin X_2^j \cup X_3^j$, for $j = 1, 2, 3$. Since there is no triad in $\mathcal{V}_2 \cup \mathcal{V}_3$, it follows that $|\{y, z\} \cap \mathcal{V}_2 \cup \mathcal{V}_3| \leq 1$. As $X_1^1 \cup X_1^2 \cup X_1^3$ is a clique, we deduce that $|\{y, z\} \cap (X_1^1 \cup X_1^2 \cup X_1^3)| \leq 1$. Hence we may assume by symmetry that $y \in \mathcal{V}_2 \cup \mathcal{V}_3$ and $z \in X_1^1 \cup X_1^2 \cup X_1^3$. But $\mathcal{V}_2 \cup \mathcal{V}_3$ is complete to $X_1^1 \cup X_1^2 \cup X_1^3$, a contradiction. This proves (2).

By 3.1, every color class of a $(k-2)$-coloring of $G \backslash E$ must have a vertex in $C(E)$. By (1) and (2), it follows that color classes with vertices in $A_V(E) \cup M_V(E)$ have size 2. Hence we deduce that $A_V(E) + M_V(E) + \frac{1}{2}A_W(E) + \frac{1}{2}M_W(E) \leq C(E) - 2$. Summing this inequality on all baces $E = \{x_i^j, x_{i'}^{j'}\}$ $i, j = 1, 2, 3$, it follows that

$$3\sum_i(|\mathcal{V}_i|+|\mathcal{V}^i|)+6\sum_i(|\mathcal{V}_i|+|\mathcal{V}^i|)+\frac{4}{2}\sum_{i,j}|X_i^j|+\frac{8}{2}\sum_{i,j}|X_i^j| < 9\sum_i(|\mathcal{V}_i|+|\mathcal{V}^i|)+6\sum_{i,j}|X_i^j|,$$

which is a contradiction. This proves 6.5. □

6.6. *Let $(H, H_1, H_2, H_3)^c$ be a path of triangle and (I, I_1, I_2, I_3) an antiprismatic three-cliqued graph. Let G be a worn hex-join of (H, H_1, H_2, H_3) and (I, I_1, I_2, I_3), and G' be a reduced thickening of (G, F) for some valid $F \in V(G)^2$ such that $\chi(G') > \omega(G')$. Then there exists a Tihany clique K in G', with $|K| \leq 4$.*

Proof. Assume not. Let the set X_j of H be as in the definition of a path of triangle and we may assume that $H_i = \cup_{j=i \mod 3} X_j$.

Assume first that $|\hat{X}_{2i}| > 1$ for some i. Let $u \in X_{2i-2}$ and $v \in X_{2i+2}$, so uv is an edge in G. Moreover $\{u, v\}$ is in the strong core. Thus

$$C_G(\{u, v\}) = \bigcup_{\substack{j = 2i+2 \mod 3, \\ j \geq 2i+2}} X_j \cup \bigcup_{\substack{j = 2i-2 \mod 3, \\ j \leq 2i-2}} X_j \cup I_k$$

for $k = 2i+1 \mod 3$. Hence $C_G(\{u,v\})$ is a clique and so by 3.6, there is a Tihany brace in G', a contradiction. Hence we may assume that $|\hat{X}_{2i}| = 1\ \forall i$.

Assume that $n \geq 3$ and let $u \in \hat{X}_2, v \in \hat{X}_6$. Then uv is an edge in G. Moreover $\{u,v\}$ is in the strong core. Thus

$$C_G(\{u,v\}) = \bigcup_{j=0 \mod 3, j\geq 6} X_j \cup X_2 \cup R_3 \cup L_5 \cup H_3.$$

Hence $C_G(\{u,v\})$ is an antimatching, and by 3.8, there exists a Tihany triangle in G', a contradiction. It follows that $n \leq 2$.

Assume now that $n = 2$. Let $u \in \hat{X}_2, v \in L_5$. Then uv is an edge in G and $C_G(\{u,v\}) = X_2 \cup R_3 \cup L_5 \cup H_3$. Thus $G|C(\{u,v\})$ is a perfect anti-matching between R_3 and L_5. Hence by 3.8, there is a Tihany triangle in G', a contradiction.

Thus we deduce that $n = 1$. Assume that $|R_1| = |L_3| = 1$. Let $u \in X_2$ and $v \in R_1 \cup L_3$ be a neighbor of v. Without loss of generality, we may assume that $v \in L_3$. Since $C_G(\{u,v\}) \subseteq X_2 \cup L_3 \cup H_3$ is a clique, it follows by 3.6 that there is a Tihany brace in G', a contradiction. Hence we deduce that $|R_1| = |L_3| > 1$. Now, let $u \in R_1$ and $v \in L_3$ be adjacent. By 5.6, we may assume that G is not a 2-non-substantial graph. If follows that there exists $x \in I_2$ such that x is in a triad. Thus $C_G(\{u,v,x\})$ is an antimatching, and by 3.8, there exists a Tihany clique K in G' with $|K| \leq 4$, a contradiction. This proves 6.6. □

6.7. *Let (G,A,B,C) be an antiprismatic graph that admit a worn chain decomposition (G_i,A_i,B_i,C_i). Suppose that there exists k such that (G_k,A_k,B_k,C_k) is the line graph of $K_{3,3}$. Let G' be a reduced thickening of (G,F) for some valid $F \in V(G)^2$. If $\chi(G') > \omega(G')$, then there is a Tihany brace in G'.*

Proof. Assume not. Let $\{a^i_j\}_{i,j=1,2,3}$ be the vertices of G_k using the standard notation. Let $X^i_j = X_{a^i_j}$ be the clique corresponding to a^i_j in the thickening. Moreover, let $x^j_i \in X^j_i$, $w^j_i = |X^j_i|$ and $k = \chi(G)$.

Since all of the vertices in the thickening of G_k are in triads, G_k is linked to the rest of the graph by a hex-join.

Note that $G\backslash\{x^1_1, x^1_2\}$ is k-2 colourable. By 3.1, it follows that every color class containing a vertex in $X^2_1 \cup X^3_1$ must have a vertex in $X^1_2 \cup X^1_3$. Hence we deduce that $w^2_1 + w^3_1 \leq w^1_2 + w^1_3 - 1$ and by symmetry $w^2_2 + w^3_2 \leq w^1_1 + w^1_3 - 1$. Summing these two inequalities, it follows that

$$w^2_1 + w^3_1 + w^2_2 + w^3_2 < w^1_2 + w^1_1 + 2w^1_3.$$

A similar inequality can be obtained for all edges $x^j_i x^j_{i'}$. Summing them all, we deduce that $4\sum_{ij} w^j_i < 2\sum_{ij} w^j_i + 2\sum_{ij} w^j_i$, a contradiction. This proves 6.7 □

6.8. *Let H be a 3-coloured prismatic graph. Let G be a reduced thickening of $(\overline{H},F)$ for some valid $F \in V(G)^2$ such that $\chi(G) > \omega(G)$. Then there exists a Tihany brace or triangle in G.*

Proof. By 6.2, H admits a worn chain decomposition with all terms in $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$. If one term of the decomposition is in $\mathcal{Q}_2$ then by 6.6, it follows that there is a Tihany clique K with $|K| \leq 4$ G. If one term of the decomposition is in $\mathcal{Q}_1$, then by 6.7, it follows that there is a Tihany brace in G. Hence we may assume that all terms are in $\mathcal{Q}_0$. Therefore there are no triads in G and thus by 5.1, it follows that there is a Tihany brace in G. This proves 6.8. □

We can now prove the main result of this section.

6.9. *Let H be an orientable prismatic graph. Let G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \subseteq V(G)^2$ such that $\chi(G) > \omega(G)$. Then there exists a Tihany clique K in G with $|K| \leq 4$.*

Proof. If H admits a worn chain decomposition with all terms in $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$, then by 6.8, G admits a Tihany brace or triangle. Otherwise, by 6.1, H is either not 3-substantial, a cycle of triangles, a ring of five graph, or a mantled $L(K_{3,3})$.

If H is not 3-substantial, then by 5.7, there is a clique K in G with $|K| \leq 4$. If H is a cycle of triangles, then by 6.3, there is a Tihany brace or triangle in G. If H is a ring of five graph, then by 6.4, there is a Tihany triangle in G. Finally, if H is a mantled $L(K_{3,3})$, then by 6.5, there is a Tihany brace in G. This proves 6.9. □

7 Non-orientable Prismatic Graphs

The definitions needed to understand this section can be found in appendix B. The following is a result from [5].

7.1. *Let G be prismatic. Then G is orientable if and only if no induced subgraph of G is a twister or rotator.*

In the following two lemmas, we study complements of orientable prismatic graphs. We split our analysis based on whether the graph contains a twister or a rotator as an induced subgraph.

7.2. *Let H be an non-orientable prismatic graph. Assume that there exists $D \subseteq V(H)$ such that $G|D$ is a rotator. Let G be a reduced thickening of $(\overline{H}, F)$ such that $\chi(G) > \omega(G)$ for some valid $F \subseteq V(G)^2$. Then there exists a Tihany clique K in G with $|K| \leq 5$.*

Proof. Assume not. Let $D = \{v_1, \dots, v_9\}$ be as in the definition of a rotator. For $i = 1, 2, 3$, let A_i be the set of vertices of $V(H) \backslash D$ that are adjacent to v_i. Since H is prismatic and $\{v_1, v_2, v_3\}$ is a triangle, it follows that $A_1 \cup A_2 \cup A_3 = V(H) \backslash D$.

Let $I_1 = \{\{5,6\}, \{5,9\}, \{6,8\}, \{8,9\}\}$, $I_2 = \{\{4,6\}, \{4,9\}, \{6,7\}, \{7,9\}\}$ and $I_3 = \{\{4,5\}, \{4,8\}, \{5,7\}, \{7,8\}\}$. For $i = 1, 2, 3$ and $\{k, l\} \in I_i$, let $A_i^{k,l}$ be the set of vertices of $V(H) \backslash D$ that are complete to $\{v_i, v_k, v_l\}$. Since $\{v_1, v_2, v_3\}$ and $\{v_i, v_{i+3}, v_{i+6}\}$ are triangles for $i = 1, 2, 3$ and H is prismatic, we deduce that $A_i = \bigcup_{\{k,l\} \in I_i} A_i^{k,l}$ for $i = 1, 2, 3$. For $i = 1, 2, 3$ and $\{k, l\} \in I_i$ and since $\{v_1, v_4, v_7\}, \{v_2, v_5, v_8\}, \{v_3, v_6, v_9\}$

are triangles and H is prismatic, it follows that $A_i^{k,l}$ is anticomplete to v_m for all $m \in \{4,5,6,7,8,9\} \backslash \{i,k,l\}$.

Assume that A_2^{49} and A_3^{48} are not empty. Since H is prismatic, we deduce that A_2^{49} is anticomplete to A_3^{48} in H. Let $x \in A_2^{49}$ and $y \in A_3^{48}$. Then $C_{\overline{H}}(\{v_1, v_5, v_6, x, y\}$ is a clique and $\{v_1, v_5, v_6, x, y\}$ is in the strong core. Hence by 3.6, there exists a Tihany clique of size 5 in G.

Assume now that A_2^{49} is not empty, but A_3^{48} is empty. Let $x \in A_2^{49}$. Then $C_{\overline{H}}(\{v_1, v_5, v_6, x\})$ is a clique and $\{v_1, v_5, v_6, x\}$ is in the core. Moreover $\{v_1, v_6, x\}$ is in the strong core. Since $\{v_2, v_5, v_8\}$ is a triad and v_2 is in the strong core, it follows that if there exists $E \in F$ with $v_5 \in E$, then $E = \{v_5, v_8\}$. But v_8 is not adjacent to v_6 in $\overline{H}$. Hence by 3.6, there exists a Tihany clique K of size 4 in G.

We may now assume that $A_2^{49} = A_3^{48} = \emptyset$. Since H is prismatic, it follows that $C_{\overline{H}}(\{v_1, v_5, v_6\})$ is an anti-matching. Moreover $\{v_1, v_5, v_6\}$ is in the core and v_1 is in the strong core. For $i = 2, 3$, since $\{v_i, v_{i+3}, v_{i+6}\}$ is a triad and v_i is in the strong core, it follows that if there exists $E \in F$ with $v_{i+3} \in E$, then $E = \{v_{i+3}, v_{i+6}\}$. But v_8 is not adjacent to v_6 and v_9 is not adjacent to v_5. Hence by 3.6, there exists a Tihany triangle in G. This concludes the proof of 7.2. $\square$

7.3. *Let H be a non-orientable prismatic graph. Assume that there exists $W \subseteq V(H)$ such that $H|W$ is a twister. Further, assume that there is no induced rotator in H. If G is a reduced thickening of $(\overline{H}, F)$ such that $\chi(G) > \omega(G)$, then there exists a Tihany clique K in G with $|K| \leq 4$.*

Proof. Assume not. Let $W = \{v_1, v_2, \ldots, v_8, u_1, u_2\}$ be as in the definition of a twister. Throughout the proof, all addition is modulo 8. For $i = 1, \ldots, 8$, let $A_{i,i+1}$ be the set of vertices in $V \backslash W$ that are adjacent to v_i and v_{i+1} and let $B_{i,i+2}$ be the set of vertices in $V \backslash W$ that are adjacent to v_i and v_{i+2}. Moreover, let $C \subseteq V \backslash W$ be the set of vertices that are anticomplete to W. Since H is prismatic, we deduce that $\bigcup_{i=1}^{8}(A_{i,i+1} \cup B_{i,i+2}) \cup C = V \backslash W$. Moreover $A_{i,i+1}$ is complete to $\{v_i, v_{i+1}, v_{i+3}, v_{i+6}\}$ and anticomplete to $W \backslash \{v_i, v_{i+1}, v_{i+3}, v_{i+6}\}$. Since H is prismatic, it follows also that $B_{i,i+2}$ is complete to $u_{i \bmod 2}\}$ and anticomplete to $W \backslash \{v_i, v_{i+2}, u_{i \bmod 2}\}$. Moreover, C is anticomplete to $\{v_1, v_2, \ldots, v_8\}$.

(1) *There exists $i \in \{1, \ldots, 8\}$, such that $A_{i,i+1}$ and $A_{i+3,j+4}$ are either both empty or both non-empty.*

Assume not. By symmetry we may assume that $A_{1,2}$ is not empty and $A_{4,5}$ is empty. Since $A_{1,2}$ is not empty, we deduce that $A_{6,7}$ is empty. Since $A_{4,5}$ and $A_{6,7}$ are empty, it follows that $A_{7,8}$ and $A_{3,4}$ are not empty. Let $x \in A_{7,8}$ and $y \in A_{3,4}$. Then $G|\{v_8, u_1, v_4, x, v_6, v_3, v_7, v_2, y\}$ is a rotator, a contradiction. This proves (1).

(2) *If $A_{i,i+1}$ and $A_{i+3,i+4}$ are both non-empty for some $i \in \{1, \ldots, 8\}$, then there exists a Tihany clique of size 5 in G.*

Assume that $A_{2,3}$ and $A_{5,6}$ are not empty and let $x \in A_{2,3}$ and $y \in A_{5,6}$. The anti-neighborhood of $\{v_1, v_7, u_2, x, y\}$ in H is a stable set. Moreover, $\{v_1, v_7, u_2, x, y\}$ is in the strong core and hence by 3.6 there is a Tihany clique of size 5 in G. This proves (2).

(3) *If $A_{i,i+1}$ and $A_{i+3,i+4}$ are both empty for some $i \in \{1,\dots,8\}$, then there exists a Tihany clique of size 4 in G.*

Assume that $A_{2,3}$ and $A_{5,6}$ are both empty. Then the anti-neighborhood of $\{v_1, v_7, u_2\}$ in H is $A_{8,2} \cup A_{2,4} \cup A_{4,6} \cup A_{6,8}$ which is a matching. Moreover u_2 is in the strong core and $\{v_1, v_7\}$ is in the core. Possibly $\{v_1, v_5\}$ and $\{v_3, v_7\}$ are in F, but $A_{2,8} \cup A_{2,4} \cup A_{4,6} \cup A_{6,8} \cup \{v_3, v_7\}$ is also an anti-matching. Hence by 3.8, there is a Tihany clique of size 4 in G. This proves (3).

Now by (1), there exists i such that $A_{i,i+1}$ and $A_{i+3,i+4}$ are either both empty or both non-empty. If $A_{i,i+1}$ and $A_{i+3,i+4}$ are both non-empty, then by (2) there is a Tihany clique of size 5 in G. If $A_{i,i+1}$ and $A_{i+3,i+4}$ are both empty, then by (3) there is a Tihany clique of size 4 in G. This concludes the proof of 7.3. □

7.4. *Let H be a non-orientable prismatic graph. Let G be a reduced thickening of $(\overline{H}, F)$ for some valid $F \subseteq V(G)^2$ such that $\chi(G) > \omega(G)$; then there exists a Tihany clique K in G with $K \leq 5$.*

Proof. By 7.1, it follows that there is an induced twister or an induced rotator in H. If there is an induced rotator in H, then by 7.2, it follows that there is a Tihany clique of size 5 in G. If there is an induced twister and no induced rotator in H, then by 7.3, it follows that there is a Tihany clique of size 4 in G. This proves 7.4. □

8 Three-cliqued Graphs

In this section we prove Theorem 1.1 for those claw-free graphs G for which $V(G)$ can be partitioned into three cliques. The definition of three-cliqued graphs has been given at the start of Section 6. A list of three-cliqued claw-free graphs that are needed for the statement of the structure theorem can be found in appendix C. We begin with a structure theorem from [6].

8.1. *Every three-cliqued claw-free graph admits a worn hex-chain into terms each of which is a reduced thickening of a permutation of a member of one of $\mathcal{TC}_1, \dots, \mathcal{TC}_5$.*

Let (G, A, B, C) be a three-cliqued graph and K be a clique of G. We say that K is *strongly Tihany* if for all three-cliqued graphs (H, A', B', C'), K is Tihany in every worn hex-join $(I, A \cup A', B \cup B', C \cup C')$ of (G, A, B, C) and (H, A', B', C') such that $\chi(I) > \omega(I)$.

A clique K is said to be bi-cliqued if exactly two of $K \cap A, K \cap B, K \cap C$ are not empty and every $v \in K$ is in a triad. A clique K is said to be tri-cliqued if $K \cap A, K \cap B, K \cap C$ are all not empty and every $v \in K$ is in a triad.

8.2. *Let K be a dense clique in (G, A_1, A_2, A_3). If both K and $\overline{C}(K)$ are bi-cliqued, then K is strongly Tihany.*

Proof. Let (G', A', B', C') be a three-cliqued claw-free graph and let (H, D, E, F) be a worn hex-join of (G, A, B, C) and (G', A', B', C'). Then in H, $C(K) \cap V(G')$ is a clique that is complete to $C(K) \cap V(G)$. Hence, by 3.2, K is Tihany in H and hence H is strongly Tihany. $\square$

8.3. *Let K be a dense clique of a three-cliqued graph (G, A, B, C). If K is tri-cliqued, then K is strongly Tihany.*

Proof. Let (G', A', B', C') be a three-cliqued claw-free graph and let (H, D, E, F) be a hex-join of (G, A, B, C) and (G', A', B', C'). Then in H, $C_H(K) \cap V(G') = \emptyset$ and thus $C_H(K)$ is a clique in H. Hence, by 3.2, K is strongly Tihany. $\square$

8.4. *Let (G, A, B, C) be an element of $\mathcal{TC}_1$ and G' be a reduced thickening of (G, F) for some valid $F \subseteq V(G)^2$. Then there is either a strongly Tihany brace or a strongly Tihany triangle in G'.*

Proof. Let H, v_1, v_2, v_3 be as in the definition of $\mathcal{TC}_1$; so $L(H) = G$. Let V_{12} be the set of vertices of H that are adjacent to v_1 and v_2 but not to v_3 and let V_{13}, V_{23} be defined similarly. Let V_{123} be the set of vertices complete to $\{v_1, v_2, v_3\}$.

Suppose that $V_{ij} \neq \emptyset$ for some i, j. Then let $v_{ij} \in V_{ij}$, and let x_i be the vertex in G corresponding to the edge $v_{ij}v_i$ in H and x_j be the vertex in G corresponding to the edge $v_{ij}v_j$ in H. Then $C_G(\{x_i, x_j\}) = \emptyset$, and thus by 3.5 and 8.2, there exists a strongly Tihany brace in G'.

So we may assume that $V_{ij} = \emptyset$ for all i, j. Then from the definition of $\mathcal{TC}_1$, it follows that V_{123} is not empty. Let $v \in V_{123}$ and let x_1, x_2, x_3 be the vertices in G corresponding to the edges vv_1, vv_2, vv_3 of H, respectively. Then $C_G(\{x_1, x_2, x_3\}) = \emptyset$ and hence by 3.5 and 8.3, there exists a strongly Tihany triangle in G'. This proves 8.4. $\square$

8.5. *Let (G, A, B, C) be an element of $\mathcal{TC}_2$ and let (G', A', B', C') be a reduced thickening of (G, F) for some valid $F \subseteq V(G)^2$. Then there is either a strongly Tihany brace or a strongly Tihany triangle in G'.*

Proof. Let $\Sigma, F_1, \ldots, F_k, L_1, L_2, L_3$ be as in the definition of $\mathcal{TC}_2$. Without loss of generality, we may assume that A is not anticomplete to B. It follows from the definition of G that there exists F_i such that $F_i \cap A$ and $F_i \cap B$ are both not empty. Let $\{x_k, \ldots, x_l\} = V(H) \cap F_i$ and without loss of generality, we may assume that $\{x_k, \ldots, x_l\}$ are in order on Σ.

Let F_i be such that there exists no j with $F_i \subset F_j$. Let $\{x_k, \ldots, x_l\} = V(H) \cap F_i$ and without loss of generality, we may assume that $\{x_k, \ldots, x_l\}$ are in order on Σ. Since $C(\{x_k, x_l\}) = \{x_{k+1}, \ldots, x_{l-1}\}$, it follows that $\{x_k, x_l\}$ is dense. If x_k, x_l are the endpoints of F_i, it follows by 3.1 and 3.5 that there is a Tihany brace in G. Otherwise, by 3.6 there exists a Tihany brace in G. This proves 4.2. $\square$

8.6. *Let (G, A, B, C) be an element of $\mathcal{TC}_3$ and let (G', A', B', C') be a reduced thickening of (G, F) for some valid $F \in V(G)^2$. Then there is either a strongly Tihany brace or a strongly Tihany triangle in G'.*

Proof. Let $H, A = \{a_0, a_1, \dots, a_n\}, B = \{b_0, b_1, \dots, b_n\}$, $C = \{c_1, \dots, c_n\}$, and X be as in the definition of near-antiprismatic graphs. Suppose that for some i, $a_i, b_i \in V(G)$. Then since $|C \setminus X| \geq 2$, it follows that there exists $j \neq i$ such that $c_j \in V(G)$. Now $T = \{a_i, b_i, c_j\}$ is dense and tri-cliqued in G, and so by 3.5 and 8.3 there is a strongly Tihany triangle in G'.

So we may assume that for all i, if $a_i \in V(G)$, then $b_i \notin V(G)$. Since by definition of $\mathcal{TC}_3$ every vertex is in a triad, it follows that $c_i \in V(G)$ whenever $a_i \in V(G)$. Now suppose that $a_i, a_j \in V(G)$ for some $i \neq j$. Then $(\{a_i, a_j\}, \{c_i, c_j\})$ is a non-reduced homogeneous pair in G. Hence we may assume that for all $i \neq j$ at most one of a_i, a_j is in $V(G)$. Let $a_i \in V(G)$,; then for some $j \neq i$ we have $c_j \in V(G)$. Now $E = \{a_i, c_j\}$ is dense and bi-cliqued. Moreover $\overline{C}(E)$ is bi-cliqued, hence by 3.5 and 8.2, it follows that E is a strongly Tihany brace in G'. This proves 8.6. □

8.7. *Let G be an element of $\mathcal{TC}_5$ and G' be a reduced thickening of (G, F) for some valid $F \subseteq V(G)^2$. Then there exists either a brace $E \in V(G')$ that is strongly Tihany or a triangle $T \in V(G')$ that is strongly Tihany in G'.*

Proof. First suppose that $G \in \mathcal{TC}_5^1$. Let $H, \{v_1, \dots, v_8\}$ be as in the definition of $\mathcal{TC}_5^1$. If $v_4 \in V(G)$ then $\{v_2, v_4\}$ is dense and bi-cliqued. Moreover $\overline{C}(\{v_2, v_4\})$ is bi-cliqued and thus by 3.5 and 8.2, there is a strongly Tihany brace in G'. If $v_3 \in G$, then $\{v_3, v_5\}$ is dense and bi-cliqued. Moreover $\overline{C}(\{v_3, v_5\})$ is bi-cliqued and so by 3.5 and 8.2, there is a strongly Tihany brace in G'. So we may assume that $v_4, v_3 \notin V(G)$. But then the triangle $T = \{v_1, v_6, v_7\}$ is dense and tri-cliqued and thus by 3.5 and 8.3, there exists a strongly Tihany triangle in G'.

We may assume now that $G \in \mathcal{TC}_5^2$. If $v_3 \in G$ then $\{v_2, v_3\}$ is dense, bi-cliqued and $\overline{C}(\{v_2, v_3\})$ is bi-cliqued. Otherwise, $\{v_2, v_4\}$ is dense, bi-cliqued and $\overline{C}(\{v_2, v_4\})$ is bi-cliqued. In both cases, it follows from 3.5 and 8.2 that there exists a strongly Tihany brace in G'. This proves 8.7. □

We are now ready to prove the main result of this section.

8.8. *Let G be a three-cliqued claw-free graph such that $\chi(G) > \omega(G)$. Then G contains either a Tihany brace or a Tihany triangle in G.*

Proof. By 8.1, there exist (G_i, A_i, B_i, C_i), for $i = 1, \dots, n$, such that the sequence (G_i, A_i, B_i, C_i) $(i = 1, \dots, n)$ is a worn hex-chain for (G, A, B, C) and such that (G_i, A_i, B_i, C_i) is a reduced thickening of a permutation of a member of one of $\mathcal{TC}_1, \dots, \mathcal{TC}_5$. If there exists $i \in \{1, \dots, n\}$ such that (G_i, A_i, B_i, C_i) is a reduced thickening of a permutation of a member of $\mathcal{TC}_1$, $\mathcal{TC}_2$, $\mathcal{TC}_3$, or $\mathcal{TC}_5$, then by 8.4, 8.5, 8.6, or 8.7 (respectively), there is a strongly Tihany brace or a strongly Tihany triangle in G_i, and thus there is a Tihany brace or a Tihany triangle in G. Thus it follows that (G_i, A_i, B_i, C_i) is a reduced thickening of a member of $\mathcal{TC}_4$ for all $i = 1, \dots, n$. Hence G is a reduced thickening of a three-cliqued antiprismatic graph. By 6.8, there exists a Tihany brace or triangle in G. This proves 8.8 □

9 Non-trivial Strip Structures

In this section we prove 1.1 for graphs G that admit non-trivial strip structures and appear in [6].

Let (J, Z) be a strip. We say that (J, Z) is a *line graph strip* if $|V(J)| = 3$, $|Z| = 2$ and Z is complete to $V(J) \setminus Z$.

The following two lemmas appear in [3].

9.1. *Suppose that G admits a nontrivial strip-structure such that $|Z| = 1$ for some strip (J, Z) of (H, η). Then either G is a clique or G admits a clique cutset.*

9.2. *Let G be a graph that admits a nontrivial strip-structure (H, η) such that for every $F \in E(H)$, the strip of (H, η) at F is a line graph strip. Then G is a line graph.*

We now use these lemmas to prove the main result of this section.

9.3. *Let G be a claw-free graph with $\chi(G) > \omega(G)$ that is a minimal counterexample to 1.1. Then G does not admit a nontrivial strip-structure (H, η) such that for each strip (J, Z) of (H, η), $1 \leq |Z| \leq 2$, and if $|Z| = 2$ then either $|V(J)| = 3$ and Z is complete to $V(J) \setminus Z$, or (J, Z) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$.*

Proof. Suppose that G admits a nontrivial strip-structure (H, η) such that for each strip (J, Z) of (H, η), $1 \leq |Z| \leq 2$. Further suppose that $|Z| = 1$ for some strip (J, Z). Then by 9.1 either G is a clique or G admits a clique cutset; in the former case G does not satisfy $\chi(G) > \omega(G)$, and in the latter case 9.3 follows from 3.10. Hence we may assume that $|Z| = 2$ for all strips (J, Z).

If all the strips of (H, η) are line graph strips, then by 9.2, G is a line graph and the result follows from [1]. So we may assume that some strip (J_1, Z_1) is not a line graph strip. Let $Z_1 = \{a_1, b_1\}$. Let $A_1 = N_{J_1}(a_1)$, $B_1 = N_{J_1}(b_1)$, $A_2 = N_G(A_1) \setminus V(J_1)$, and $B_2 = N_G(B_1) \setminus V(J_1)$. Let $C_1 = V(J_1) \setminus (A_1 \cup B_1)$ and $C_2 = V(G) \setminus (V(J_1) \cup A_2 \cup B_2)$. Then $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2 \cup B_2 \cup C_2$.

(1) *If $C_2 = \emptyset$ and $A_2 = B_2$, then there is a Tihany clique K in G with $|K| \leq 5$.*

Note that $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2$. Since $|Z_1| = 2$ and (J_1, Z_1) is not a line graph strip, it follows that (J_1, Z_1) is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$. We consider the cases separately:

1. If (J_1, Z_1) is a member of $\mathcal{Z}_1$, then J_1 is a fuzzy linear interval graph and so G is a fuzzy long circular interval graph and Theorem 9.3 follows from [1].

2. If (J_1, Z_1) is a member of $\mathcal{Z}_2, \mathcal{Z}_3$, or $\mathcal{Z}_4$. In all of these cases, A_1, B_1, and C_1 are all cliques and so $V(G)$ is the union of three cliques, namely $A_1 \cup A_2, B_1$, and C_1. Hence, by 8.8, there exists a Tihany clique K with $|K| \leq 5$.

3. If (J_1, Z_1) is a member of $\mathcal{Z}_5$. Let $v_1, \ldots, v_{12}, X, H, H', F$ be as in the definition of $\mathcal{Z}_5$ and for $1 \leq i \leq 12$ let X_{v_i} be as in the definition of a thickening. Then A_2 is complete to $X_{v_1} \cup X_{v_2} \cup X_{v_4} \cup X_{v_5}$. Let H'' be the graph obtained from

H' by adding a new vertex a_2, adjacent to v_1, v_2, v_4 and v_5. Then H'' is an antiprismatic graph. Moreover, no triad of H'' contains v_9 or v_{10}. Thus the pair (H', F) is antiprismatic, and G is a thickening of (H', F), so 9.3 follows from 6.9 and 7.4.

This proves (1).

By (1), we may assume that either $C_2 \neq \emptyset$, or $A_2 \neq B_2$. Suppose that $A_2 = B_2$. Then since $C_2 \neq \emptyset$ it follows that A_2 is a clique cutset of G and the result follows from 3.10. Hence, we may assume that $A_2 \neq B_2$ and without loss of generality we may assume that $A_2 \setminus B_2 \neq \emptyset$. Let $v \in A_2 \setminus B_2$ and let $w \in A_1 \setminus B_1$. Then $E = \{v, w\}$ is dense and 9.3 follows from 3.2. □

10 Proof of the Main Theorem

We can now prove the main theorem.

Proof of 1.1. Let G be a claw-free graph with $\chi(G) > \omega(G)$and suppose that there does not exist a clique K in G with $|K| \leq 5$ such that $\chi(G \backslash K) > \chi(G) - |K|$. By 9.3 and 2.1, it follows that either G is a member of $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ or $V(G)$ is the union of three cliques. By 4.1, it follows that G is not a member of $\mathcal{T}_1$. By 4.2, it follows that G is not a member of $\mathcal{T}_2$. By 6.9 and 7.4, we deduce that G is not a member $\mathcal{T}_3$. Hence, it follows that $V(G)$ is the union of three cliques. But by 8.8, it follows that there is a Tihany brace or triangle in G, a contradiction. This proves 1.1. □

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A Orientable prismatic graphs

- $\mathcal{Q}_0$ is the class of all 3-coloured graphs (G, A, B, C) such that G has no triangle.
- $\mathcal{Q}_1$ is the class of all 3-coloured graphs (G, A, B, C) where G is isomorphic to the line graph of $K_{3,3}$.
- $\mathcal{Q}_2$ is the class of all canonically-coloured path of triangles graphs.
- **Path of triangles.** A graph G is a *path of triangles* graph if for some integer $n \geq 1$ there are pairwise disjoint stable subsets $X_1, \ldots, X_{2n+1}$ of $V(G)$ with union $V(G)$, satisfying the following conditions (P1)-(P7).

(P1) For $1 \leq i \leq n$, there is a nonempty subset $\hat{X}_{2i} \subseteq X_{2i}; |\hat{X}_2| = |\hat{X}_{2n}| = 1$, and for $0 < i < n$, at least one of $\hat{X}_{2i}, \hat{X}_{2i+2}$ has cardinality 1.

(P2) For $1 \leq i < j \leq 2n+1$

  (1) if $j - i = 2$ modulo 3 and there exist $u \in X_i$ and $v \in X_j$, nonadjacent, then either i, j are odd and $j = i + 2$, or i, j are even and $u \notin \hat{X}_i$ and $v \notin \hat{X}_j$;

  (2) if $j - i \neq 2$ modulo 3 then either $j = i + 1$ or X_i is anticomplete to X_j.

(P3) For $1 \leq i \leq n+1$, X_{2i-1} is the union of three pairwise disjoint sets $L_{2i-1}, M2i-1, R2i-1$, where $L_1 = M_1 = M_{2n+1} = R_{2n+1} = \emptyset$.

(P4) If $R_1 = \emptyset$ then $n \geq 2$ and $|\hat{X}_4| > 1$, and if $L_{2n+1} = \emptyset$ then $n \geq 2$ and $|\hat{X}_{2n-2}| > 1$.

(P5) For $1 \leq i \leq n$, X_{2i} is anticomplete to $L_{2i-1} \cup R_{2i+1}$; $X_{2i} \backslash \hat{X}_{2i}$ is anticomplete to $M_{2i-1} \cup M_{2i+1}$; and every vertex in $X_{2i} \backslash \hat{X}_{2i}$ is adjacent to exactly one end of every edge between R_{2i-1} and L_{2i+1}.

(P6) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| = 1$, then

  (1) R_{2i-1}, L_{2i+1} are matched, and every edge between $M_{2i-1} \cup R_{2i-1}$ and $L_{2i+1} \cup M_{2i+1}$ is between R_{2i-1} and L_{2i+1};

  (2) the vertex in $\hat{X}_{2i}$ is complete to $R_{2i-1} \cup M_{2i-1} \cup L_{2i+1} \cup M_{2i+1}$;

  (3) L_{2i-1} is complete to X_{2i+1} and X_{2i-1} is complete to R_{2i+1}

  (4) if $i > 1$ then $M_{2i-1}, \hat{X}_{2i-2}$ are matched, and if $i < n$ then $M_{2i+1}, \hat{X}_{2i+2}$ are matched.

(P7) For $1 < i < n$, if $|\hat{X}_{2i}| > 1$ then

  (1) $R_{2i-1} = L_{2i+1} = \emptyset$;

  (2) if $u \in X_{2i-1}$ and $v \in X_{2i+1}$, then u, v are nonadjacent if and only if they have the same neighbour in $\hat{X}_{2i}$.

Let $A_k = \bigcup(X_i : 1 \leq i \leq 2n+1 \text{ and } i = k \mod 3)$ $(k = 0, 1, 2)$. Then (G, A_1, A_2, A_3) is a *canonically-coloured path of triangles graphs.*

- **Cycle of triangles.** A graph G is a *cycle of triangles* graph if for some integer $n \geq 5$ with $n = 2$ modulo 3, there are pairwise disjoint stable subsets $X_1, \dots, X_{2n}$ of $V(G)$ with union $V(G)$, satisfying the following conditions (C1)-(C6) (reading subscripts modulo $2n$):

  (C1) For $1 \leq i \leq n$, there is a nonempty subset $\hat{X}_{2i} \subseteq X_{2i}$, and at least one of $\hat{X}_{2i}, \hat{X}_{2i+2}$ has cardinality 1.

  (C2) For $i \in \{1, \dots, 2n\}$ and all k with $2 \leq k \leq 2n-2$, let $j \in \{1, \dots, 2n\}$ with $j = i + k$ modulo 2n:

  (1) if $k = 2$ modulo 3 and there exist $u \in X_i$ and $v \in X_j$, nonadjacent, then either i, j are odd and $k \in \{2, 2n-2\}$, or i, j are even and $u \notin \hat{X}_i$ and $v \notin \hat{X}_j$;

  (2) if $k \neq 2$ modulo 3 then X_i is anticomplete to X_j.

  (Note that $k = 2$ modulo 3 if and only if $2n - k = 2$ modulo 3, so these statements are symmetric between i and j.)

  (C3) For $1 \leq i \leq n+1$, X_{2i-1} is the union of three pairwise disjoint sets $L_{2i-1}, M_{2i-1}, R_{2i-1}$.

  (C4) For $1 \leq i \leq n$, X_{2i} is anticomplete to $L_{2i-1} \cup R_{2i+1}$; $X_{2i} \backslash \hat{X}_{2i}$ is anticomplete to $M_{2i-1} \cup M_{2i+1}$; and every vertex in $X_{2i} \backslash \hat{X}_{2i}$ is adjacent to exactly one end of every edge between R_{2i-1} and L_{2i+1}.

  (C5) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| = 1$, then

  (1) R_{2i-1}, L_{2i+1} are matched, and every edge between $M_{2i-1} \cup R_{2i-1}$ and $L_{2i+1} \cup M_{2i+1}$ is between R_{2i-1} and L_{2i+1};

  (2) the vertex in $\hat{X}_{2i}$ is complete to $R_{2i-1} \cup M_{2i-1} \cup L_{2i+1} \cup M_{2i+1}$;

  (3) L_{2i-1} is complete to X_{2i+1} and X_{2i-1} is complete to R_{2i+1}

  (4) $M_{2i-1}, \hat{X}_{2i-2}$ are matched and $M_{2i+1}, \hat{X}_{2i+2}$ are matched.

  (C6) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| > 1$ then

  (1) $R_{2i-1} = L_{2i+1} = \emptyset$;

  (2) if $u \in X_{2i-1}$ and $v \in X_{2i+1}$, then u, v are nonadjacent if and only if they have the same neighbour in $\hat{X}_{2i}$.

- **Ring of five.** Let G be a graph with $V(G)$ the union of the disjoint sets $W = \{a_1, \dots, a_5, b_1, \dots, b_5\}$ and $V_0, V_1, \dots, V_5$. Let adjacency be as follows (reading subscripts modulo 5). For $1 \leq i \leq 5$, $\{a_i, a_{i+1}; b_{i+3}\}$ is a triangle, and a_i is adjacent to b_i; V_0 is complete to $\{b_1, \dots, b_5\}$ and anticomplete to $\{a_1, \dots, a_5\}$; $V_0, V_1, \dots, V_5$ are all stable; for $i = 1, \dots, 5$, V_i is complete to $\{a_{i-1}, b_i, a_{i+1}\}$ and anticomplete to the remainder of W; V_0 is anticomplete to $V_1 \cup \dots \cup V_5$; for $1 \leq i \leq 5$ V_i is anticomplete to V_{i+2}; and the adjacency between V_i, V_{i+1} is arbitrary. We call such a graph a *ring of five*.

- **Mantled $L(K_{3,3})$.** Let G be a graph with $V(G)$ the union of seven sets

$$W = \{a_i^j : 1 \leq i, j \leq 3\}, V^1, V^2, V^3, V_1, V_2, V_3,$$

with adjacency as follows. For $1 \le i, j, i', j' \le 3$, a_i^j and $a_{i'}^{j'}$ are adjacent if and only if $i' \neq i$ and $j' \neq j$. For $i = 1, 2, 3, V^i, V_i$ are stable; V^i is complete to $\{a_i^1, a_i^2, a_i^3\}$, and anticomplete to the remainder of W; and V_i is complete to $\{a_1^i, a_2^i, a_3^i\}$ and anticomplete to the remainder of W. Moreover, $V^1 \cup V^2 \cup V^3$ is anticomplete to $V_1 \cup V_2 \cup V_3$, and there is no triangle included in $V^1 \cup V^2 \cup V^3$ or in $V_1 \cup V_2 \cup V_3$. We call such a graph G a *mantled* $L(K_{3,3})$.

B Non-orientable prismatic graphs

- **A rotator**. Let G have nine vertices $v_1, v_2, \dots, v_9$, where $\{v_1, v_2, v_3\}$ is a triangle, $\{v_4, v_5, v_6\}$ is complete to $\{v_7, v_8, v_9\}$, and for $i = 1, 2, 3$, v_i is adjacent to v_{i+3}, v_{i+6}, and there are no other edges. We call G a *rotator*.

- **A twister**. Let G have ten vertices $u_1, u_2, v_1, \dots, v_8$, where u_1, u_2 are adjacent, for $i = 1, \dots, 8$ v_i is adjacent to $v_{i-1}, v_{i+1}, v_{i+4}$ (reading subscripts modulo 8), and for $i = 1, 2$, u_i is adjacent to $v_i, v_{i+2}, v_{i+4}, v_{i+6}$, and there are no other edges. We call G a *twister* and u_1, u_2 is the *axis* of the twister.

C Three-cliqued graphs

- **A type of line trigraph**. Let v_1, v_2, v_3 be distinct nonadjacent vertices of a graph H, such that every edge of H is incident with one of v_1, v_2, v_3. Let v_1, v_2, v_3 all have degree at least three, and let all other vertices of H have degree at least one. Moreover, for all distinct $i, j \in \{1, 2, 3\}$, let there be at most one vertex different from v_1, v_2, v_3 that is adjacent to v_i and not to v_j in H. Let A, B, C be the sets of edges of H incident with v_1, v_2, v_3 respectively, and let G be a line trigraph of H. Then (G, A, B, C) is a three-cliqued claw-free trigraph; let $\mathcal{TC}_1$ be the class of all such three-cliqued trigraphs such that every vertex is in a triad.

- **Long circular interval trigraphs**. Let G be a long circular interval trigraph, and let Σ be a circle with $V(G) \subseteq \Sigma$, and $F_1, \dots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(G)$ with $|X| \le 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(G)$. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(G) \subseteq L_1 \cup L_2 \cup L_3$; then $(G, V(G) \cap L_1, V(G) \cap L_2, V(G) \cap L_3)$ is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_2$ the class of such three-cliqued trigraphs with the additional property that every vertex is in a triad.

- **Near-antiprismatic trigraphs**. Let H be a near-antiprismatic trigraph, and let A, B, C, X be as in the deffnition of near-antiprismatic trigraph. Let $A' = A \backslash X$ and define B', C' similarly; then (H, A', B', C') is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_3$ the class of all three-cliqued trigraphs with the additional property that every vertex is in a triad.

- **Antiprismatic trigraphs**. Let G be an antiprismatic trigraph and let A, B, C be a partition of $V(G)$ into three strong cliques; then (G, A, B, C) is a three-cliqued claw-free trigraph. We denote the class of all such three-cliqued trigraphs by $\mathcal{TC}_4$. (In [4] Chudnovsky and Seymour described explicitly all three-cliqued antiprismatic graphs, and their "changeable" edges; and this therefore provides a description of the three-cliqued antiprismatic trigraphs.) Note that in this case there may be vertices that are in no triads.

- **Sporadic exceptions.**

  - Let H be the trigraph with vertex set $\{v_1, \dots, v_8\}$ and adjacency as follows: v_i, v_j are strongly adjacent for $1 \leq i < j \leq 6$ with $j - i \leq 2$; the pairs v_1v_5 and v_2v_6 are strongly antiadjacent; v_1, v_6, v_7 are pairwise strongly adjacent, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \dots, v_6$; the pairs v_1v_4 and v_3v_6 are semiadjacent, and v_2 is antiadjacent to v_5. Let $A = \{v_1, v_2, v_3\}, B = \{v_4, v_5, v_6\}$ and $C = \{v_7, v_8\}$. Let $X \subseteq \{v_3, v_4\}$; then $(H \backslash X, A \backslash X, B \backslash X, C)$ is a three-cliqued claw-free trigraph, and all its vertices are in triads.
  - Let H be the trigraph with vertex set $\{v_1, \dots, v_9\}$, and adjacency as follows: the sets $A = \{v_1, v_2\}$, $B = \{v_3, v_4, v_5, v_6, v_9\}$ and $C = \{v_7, v_8\}$ are strong cliques; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and the adjacency between the pairs v_2v_4 and v_5v_7 is arbitrary. Let $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that
    - v_2 is not strongly anticomplete to $\{v_3, v_4\} \backslash X$
    - v_7 is not strongly anticomplete to $\{v_5, v_6\} \backslash X$
    - if $v_4, v_5 \notin X$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

    Then $(H \backslash X, A, B \backslash X, C)$ is a three-cliqued claw-free trigraph.

  We denote by $\mathcal{TC}_5$ the class of such three-cliqued trigraphs (given by one of these two constructions) with the additional property that every vertex is in a triad.

D Strips

$\mathcal{Z}_1$: Let H be a graph with vertex set $\{v_1, \dots, v_n\}$, such that for $1 \leq i < j < k \leq n$, if v_i, v_k are adjacent then v_j is adjacent to both v_i, v_k. Let $n \geq 2$, let v_1, v_n be nonadjacent, and let there be no vertex adjacent to both v_1 and v_n. Let $F' \subseteq V(H)^2$ be the set of pairs $\{v_i, v_j\}$ such that $i < j$, $v_i \neq v_1$ and $v_j \neq v_n$, v_i is nonadjacent to v_{j+1}, and v_j is nonadjacent to

v_{i-1}. Furthermore, let $F \subseteq F'$ such that every vertex of H appears in at most one member of F. Then G is a *fuzzy linear interval graph* if for some H and F, G is a thickening of (H, F) with $|X_{v_1}| = |X_{v_n}| = 1$. Let $X_{v_1} = \{u_1\}$, $X_{v_n} = \{u_n\}$, and $Z = \{u_1, u_n\}$. $\mathcal{Z}_1$ is the set of all pairs (G, Z).

$\mathcal{Z}_2$: Let $n \geq 2$. Construct a graph H as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n+1$ and $|C| = n$, say $A = \{a_0, a_1, \dots, a_n\}, B = \{b_0, b_1, \dots, b_n\}$, and $C = \{c_1, \dots, c_n\}$. Adjacency is as follows. A, B, C are cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$, let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. All other pairs not specified so far are nonadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let $H' = H \setminus X$ and let G be a thickening of (H', F) with $|X_{a_0}| = |X_{b_0}| = 1$ and $F \subseteq V(H')^2$ (we will not specify the possibilities for the set F because they are technical and we will not need them in our proof). Let $X_{a_0} = \{a'_0\}$, $X_{b_0} = \{b'_0\}$, and $Z = \{a'_0, b'_0\}$. $\mathcal{Z}_2$ is the set of all pairs (G, Z).

$\mathcal{Z}_3$: Let H be a graph, and let h_1-h_2-h_3-h_4-h_5 be the vertices of a path of H in order, such that h_1, h_5 both have degree one in H, and every edge of H is incident with one of h_2, h_3, h_4. Let H' be obtained from the line graph of H by making the edges h_2h_3 and h_3h_4 of H (vertices of H') nonadjacent. Let $F \subseteq \{\{h_2h_3, h_3h_4\}\}$ and let G be a thickening of (H', F) with $|X_{h_1h_2}| = |X_{h_4h_5}| = 1$. Let $X_{h_1h_2} = \{u\}$, $X_{h_4h_5} = \{v\}$, and $Z = \{u, v\}$. $\mathcal{Z}_3$ is the set of all pairs (G, Z).

$\mathcal{Z}_4$: Let H be the graph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ and adjacency as follows: $\{a_0, a_1, a_2\}, \{b_0, b_1, b_2, b_3\}, \{a_2, c_1, c_2\}$, and $\{a_1, b_1, c_2\}$ are cliques; b_2, c_1 are adjacent; and all other pairs are nonadjacent. Let $F = \{\{b_2, c_2\}, \{b_3, c_1\}\}$ and let G be a thickening of (H, F) with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{a_0} = \{a'_0\}$, $X_{b_0} = \{b'_0\}$, and $Z = \{a'_0, b'_0\}$. $\mathcal{Z}_4$ is the set of all pairs (G, Z).

$\mathcal{Z}_5$: Let H be the graph with vertex set $\{v_1, \dots, v_{12}\}$, and with adjacency as follows. v_1-$\cdots$-v_6-v_1 is an induced cycle in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; and v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$. No other pairs are adjacent. Let H' be a graph isomorphic to $H \setminus X$ for some $X \subseteq \{v_{11}, v_{12}\}$ and let $F \subseteq \{\{v_9, v_{10}\}\}$. Let G be a thickening of (H', F) with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{v_7} = \{v'_7\}$, $X_{v_8} = \{v'_8\}$, and $Z = \{v'_7, v'_8\}$. $\mathcal{Z}_5$ is the set of all pairs (G, Z).