IEOR 6711: Stochastic Models I

Fall 2003, Professor Whitt

Answers for Discussion Topics on Thursday, September 18, 2003

The Poisson Process

1. **The AAAA Insurance Company**

The Athanassoglou-Asavanunt-Avdeev-Avdis (*AAAA*) Insurance Company provides automobile and homeowner's insurance. Suppose that claims arrive according to a Poisson process at a rate of 3 per day. Suppose that 2/3 of the claims are for automobile insurance, while 1/3 are for homeowners' insurance.

(a) Where does the *AAAA* Insurance Company appear in the telephone book listings for insurance companies?

Naturally, Stergios, Attakrit, Egor and Efstathios hope their listing will appear first, but they could be foiled by other clever entrepreneurs (*A* Insurance Company, *A*1 Insurance Company or *AAAAAAAA* Insurance Company, depending on the listing order).

(b) What is the probability that exactly three automobile claims arrive on a given day?

By the thinning property in Section 3.4, the arrival process of automobile insurance claims is a Poisson process with rate $(2/3)\times 3 = 2$ per day. Let $N_A(t)$ denote the number of automobile claims to arrive in t days. The desired probability is

$$P(N_A(1) = 3) = \frac{e^{-(2\times 1)}(2\times 1)^3}{3!} = \frac{8e^{-2}}{6} = 0.1804 \ .$$

(c) What is the conditional expected number of automobile insurance claims in a two-day period given that there were 10 automobile insurance claims during the *previous* two days?

A Poisson process has independent increments. Hence the conditioning does not affect the expectation. The mean number of events in an interval of length t for a Poisson process with rate λ is λt. Thus, for any t,

$$E[N_A(t+2) - N_A(t) | N_A(t) - N_A(t-2) = 10] = E[N_A(t+2) - N_A(t)] = E[N_A(2)] = 2\times 2 = 4 \ .$$

(d) What is the conditional expected *total* number of insurance claims in a four-day period (automobile plus homeowners') given that there are exactly 20 automobile insurance claims during these four days?

The conditioning fully determines the number of automobile insurance claims, but has no effect on the homeowner's insurance claims, because the two component processes are independent Poisson processes. Hence,

$$E[N_A(4) + N_H(4)|N_A(4) = 20] = 20 + E[N_H(4)|N_A(4) = 20] = 20 + E[N_H(4)] = 20 + 4 = 24.$$

(e) What is the conditional expected *total* number of insurance claims in a four-day period (automobile plus homeowners') given that there are exactly 5 automobile insurance claims during the first two of these four days?

The conditioning fully determines the number of automobile insurance claims during the first two days, but not for the second two days. The conditioning has no effect on the homeowner's insurance claims. Hence,

$$\begin{aligned} E[N_A(4) + N_H(4)|N_A(2) = 5] &= 5 + E[N_A(4) - N_A(2)|N_A(2) = 5] + E[N_H(4)|N_A(2) = 5] \\ &= 5 + E[N_A(4) - N_A(2)] + E[N_H(4)] \\ &= 5 + E[N_A(2)] + E[N_H(4)] \\ &= 5 + (2 \times 2) + (1 \times 4) = 5 + 4 + 4 = 13\ . \end{aligned}$$

Suppose that the means and standard deviations of the dollar values for individual claims of these two kinds are as given in the following table:

	mean	standard deviation
auto	\$4000	\$1000
homeowner's	\$3000	\$3000

(f) What is the expected dollar value of all insurance claims each day?

Here we have compound Poisson processes, as in Section 2.5. The formulas for means and variances are given on top of p. 83. Here we have two independent compound Poisson processes. Let λ_A and λ_H be the daily rates of the two kinds of claims. Then the expected dollar value of all insurance claims each day, measured in thousands of dollars, is

$$(\lambda_A \times 4) + (\lambda_H \times 3) = (2 \times 4) + (1 \times 3) = 11\ .$$

(g) What is the variance of the dollar value of all insurance claims each day?

Since the compound Poisson processes are independent, the variance of the sum is the sum of the variances. We thus need to compute the variance for each compound Poisson process. We use the formula on p. 83: $Var(S_N) = \lambda E[X^2]$, where N is Poisson. (The key underlying result is the conditional variance formula in Problem 1.22.) Recall that $E[X^2] = Var(X) + (EX)^2$.

Here for the automobile insurance claims and homeowner's insurance claims (now in units of 10^6),

$$E[X_A^2] = (1)^2 + (4)^2 = 17 \ ,$$

and

$$E[X_H^2] = (3)^2 + (3)^2 = 18 \ .$$

Hence, the total variance (again, in units of 10^6) is

$$\text{total variance} = \lambda_A E[X_A^2] + \lambda_H E[X_H^2] = 2(17) + 1(18) = 52 \ .$$

Thus the full variance is 5.2×10^7. The standard deviation is thus $\sqrt{52 \times 10^6} \approx 7.21 \times 10^3$, i.e, $\$7,210$.

(h) Give an expression approximating the probability that the *total* dollar value of all claims over a *ten-day* period exceeds $\$150,000$.

From the last problem, the mean and standard deviation of the total dollar value of all claims over the ten-day period are $\$110,000$ and $\$\sqrt{7 \times 10^8} \approx 2.645 \times 10^4 = \$26,450$. Since the compound Poisson process has stationary and independent increments, the distribution of the total dollar value approaches the normal distribution (with the same mean and variance) as the time interval increases, by virtue of the central limit theorem. (The stationary and independent increments property means that we can express the total dollar value as the sum of IID random variables with finite mean and variance.) Hence, the approximate probability distribution is normal. Specifically,

$$\begin{aligned} P(N(11 \times 10^4, 7 \times 10^8) > 15 \times 10^4) &= P(N(0,1) > (15.0 - 11.0)/2.645) \\ &= P(N(0,1) > 1.51) \approx 0.065 \ . \end{aligned}$$

(The desired answer is $P(N(0,1) > 4/\sqrt{7})$; that can be determined directly without doing any detailed numerical calculations.)

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2. **Gone Fishing**

After an intense fall semester, Egor has decided to take a break and go fishing in Venice, Florida, which is centrally located on the Gulf of Mexico along the coast of southwest Florida. As usual, Egor hopes to catch some grouper and snapper, but he is also hoping to catch some other fish, such as kingfish, cobia, black-fin tuna, Greater amberjack, Spanish mackerel, dolphin (mahi-mahi), shark, barracuda, tarpon, permit, little tunny, sheepshead, flounder, snook, redfish, and sea trout.

Suppose that Egor catches fish according to a Poisson process at a rate of 3 per hour.

(a) What is the expected time until Egor catches his fourth fish?

The time interval until Egor catches the first fish and the subsequent time intervals between successive fish caught are mutually independent exponential random variables with mean $(1/\lambda) = (1/3)$ hour. Hence, the expected time until Egor catches his fourth fish is $4 \times (1/3) = 4/3$ hours.

(b) What is the probability that Egor catches exactly 4 fish in a given 2-hour period?

Let $N(t)$ be the number of fish Egor catches in an interval of length t. The random variable $N(t)$ has a Poisson distribution with mean λt. Hence,

$$P(N(2) = 4) = \frac{e^{-3(2)}(6)^4}{4!} = \frac{e^{-6}(6)^4}{24} = 0.134$$

It would suffice to stop at the second expression.

(c) Suppose that Egor does indeed catch exactly 4 fish in a given 2-hour period. What then is the probability that he catches all four fish in the first 30 minutes?

Conditional on $N(t) = k$, the k arrival times are distributed in the interval $[0, t]$ as mutually independent uniform random variables. The probability that any one is in the first 30 minutes is thus $1/4$. The probability that all four are in the first 30 minutes is thus $(1/4)^4 = 1/256 = 0.0039$.

(d) According to the probability model, what is the *conditional* probability that Egor catches exactly 4 fish in a given 2-hour period, given that he catches 23 fish in the previous two hours?

According to the probability model, the Poisson process has independent increments, so that the conditioning event does not alter the probability of interest:

$$\begin{aligned} P(N(t+2) - N(t) = 4 | N(t) - N(t-2) = 23) &= P(N(t+2) - N(t) = 4) \\ &= P(N(2) = 4) = \frac{e^{-3(2)}(6)^4}{4!} = 0.134 \ . \end{aligned}$$

(e) Given your answer to part (d), what feature of the probability model would you question? How could you test whether or not that model feature is reasonable for this kind of fishing?

It is reasonable to question whether the independent increments property holds for fishing. When it is a good day for fishing, it may be good over several consecutive hours. We might have a conditional Poisson process, as in Section 2.6 of Ross.

We could test whether the events over consecutive time periods are indeed independent. We want to know if

$$P(N(t) - N(t-s) \in A, N(t+s) - N(t) \in B) = P(N(t) - N(t-s) \in A) P(N(t+s) - N(t) \in B)$$

for arbitrary time interval s and arbitrary subsets A and B. It would suffice to let $A = B$. The general idea is to test it by seeing if the observed frequencies are approximately consistent with this relationship. To do so carefully, we should apply appropriate statistical techniques. For example, we could let $s = 2$ and $A = B = \{k : k \geq 6 = EN(2)\}$. We would then see if the frequency of the event $\{N(t)-N(t-2) \in A, N(t+2)-N(t) \in A\}$ is sufficiently greater than the product of the frequencies of the events $\{N(t)-N(t-2) \in A\}$ and $\{N(t+2)-N(t) \in A\}$, where " sufficiently greater" means that it would not likely occur by chance if the latter two events were actually independent. It is not our intent to go into the statistics in detail. Without reference to statistics, it could be compared to the independent case by simulating lots of examples of the independent case. We then could calculate how likely it would be to obtain by chance in the independent case an outcome as extreme or more extreme than the one we saw.

Suppose, in addition, that each fish Egor catches is a grouper with probability 1/4, a snapper with probability 1/3 and some other kind of fish with probability 5/12, with the successive kinds being independent random trials.

(f) What is the probability that Egor catches exactly 8 fish in a given 2-hour period, with 3 of them being grouper and 5 being snapper?

The assumptions mean that we can use the thinning property: The numbers of grouper, snapper and other fish caught become independent Poisson processes with rates $3\times(1/4) = 3/4$, $3 \times (1/3) = 1$ and $3 \times (5/12) = 5/4$, respectively. Since $t = 2$, the Poisson random variables have means 1.5, 0.5 and 2.5, respectively. Letting $N_G(t)$, $N_S(t)$ and $N_O(t)$ be the numbers of grouper, snapper and other fish caught in an interval of length t, we obtain the following expression:

$$\begin{aligned} P(N_G(2) = 3, N_S(2) = 5, N_0(2) = 0) &= P(N_G(2) = 3)P(N_S(2) = 5)P(N_0(2) = 0) \\ &= \frac{e^{-1.5}(1.5)^3}{3!}\frac{e^{-2}(2)^5}{5!}\frac{e^{-2.5}(2.5)^0}{0!} \\ &= \frac{e^{-6.0}(1.5)^3(2)^5}{3!5!} = \frac{(0.002479)(3.375)(32)}{720} = 0.000372 \end{aligned}$$

(g) What is the *conditional* probability that Egor catches 3 grouper in a given 2-hour period, given that he catches 14 snapper in the same two-hour period?

Since the two Poisson processes are independent, the conditioning event has no effect. Thus

$$P(N_G(2) = 3|N_S(2) = 14) = P(N_G(2) = 3) = \frac{e^{-1.5}(1.5)^3}{3!} = \frac{(0.223)3.375}{6} = 0.126$$
