State Dependent Control of Closed Queueing Networks with Application to Ride-Hailing*

Siddhartha Banerjee† Yash Kanoria‡ Pengyu Qian§

December 11, 2018

Abstract

Inspired by ride-hailing and bike-sharing systems, we study the design of state-dependent controls for a closed queueing network model. We focus on the assignment policy, where the platform can choose which nearby vehicle to assign to an incoming customer; if no units are available nearby, the request is dropped. The vehicle becomes available at the destination after dropping the customer. We study how to minimize the proportion of dropped requests in steady state.

We propose a family of simple state-dependent policies called Scaled MaxWeight (SMW) policies that dynamically manage the geographical distribution of supply. We prove that under the complete resource pooling (CRP) condition (analogous to the condition in Hall's marriage theorem), each SMW policy leads to exponential decay of demand-dropping probability as the number of supply units scales to infinity. Further, there is an SMW policy that achieves the **optimal** exponent among all assignment policies, and we analytically specify this policy in terms of the customer arrival rates for all source-destination pairs. The optimal SMW policy maintains high supply levels near structurally under-supplied locations. We also propose data-driven approaches for designing SMW policies and demonstrate excellent performance in simulations based on the NYC taxi dataset.

*A preliminary version of this work appeared in ACM SIGMETRICS 2018 (Banerjee et al. 2018). That publication is an extended abstract containing only a subset of the current theoretical results, proof sketches, and no simulation experiments.

†School of ORIE, Cornell University. Email: sbanerjee@cornell.edu

‡Graduate School of Business, Columbia University. Email: ykanoria@columbia.edu

§Graduate School of Business, Columbia University. Email: PQian20@gsb.columbia.edu

Keywords: maximum weight policy; closed queueing network; control; Lyapunov function; shared transportation systems; large deviations; assignment.

1 Introduction

Recently, there is increasing interest in the control of shared transportation systems for ride-hailing such as Uber, Lyft, DiDi and Ola. These platforms are dynamic two-sided markets where customers (demands) arrive at different physical locations stochastically over time, and vehicles (supplies) circulate in the system as a result of transporting demands to their destinations. The platform's goal is to maximize throughput (proportion of demands fulfilled), revenue or other objectives by employing various control levers.

The main inefficiency in such systems comes from the *geographic mismatch* between supply units and customers: when a customer arrives, he has to be matched immediately with a nearby supply unit, otherwise the customer will abandon the request due to impatience. There are two sources of spatial supply-demand asymmetry: *structural imbalance* and *stochasticity.* In ride-hailing, the former is dominant during rush hours in the city when most of the demand pickups concentrate in a particular region of the city while dropoffs concentrate on others, otherwise the latter source often dominates, see Hall et al. (2015). We propose an assignment (scheduling) policy that *deals with both sources simultaneously*: the state dependent nature of policy allows near-optimal management of stochasticity, while the choice of policy parameters accounts for the structural imbalance provided it is not too large.

Many control levers have been proposed and analyzed in the literature. *Pricing*, for example, enjoys great popularity in both academia and industry, see, e.g., Waserhole and Jost (2016), Banerjee et al. (2016). By adjusting prices of rides, the system can indirectly re-balance supply and demand. Braverman et al. (2016) focus on *empty-vehicle routing*, i.e., sending available supply units to under-supplied locations in order to meet more demand.

In this paper, we study another important form of control, *assignment.* When a customer request comes in, the platform can decide from where to assign a supply unit, which will, in turn, influence the platform's ability to fulfill demand in future. Bike-sharing systems can also implement a similar control by suggesting to riders where to pick up (or drop off) a bike. Previous work has studied assignment decisions made in a state-independent manner — by optimizing the system's fluid limit, the platform can compute a desired probability of assigning from any compatible location when a demand arises, and realize this probability by randomization (Ozkan and Ward 2016, Banerjee et al. 2016). However, this approach requires exact knowledge of customer arrival rates (which is infeasible in practice), fails to react to stochastic variation in the system, and creates additional variance due to randomization. Although this control guarantees asymptotic optimality in the Law of Large Numbers sense, it converges only slowly to the fluid limit (Banerjee et al. 2016). To counter these issues, we study *state-dependent* assignment control.

We model the system as a *closed queueing network* with n servers representing physical locations, and K "jobs" that stand for supply units. After a supply unit picks up a customer, it drops her at the destination and becomes available again. (Supplies do not enter or leave the system.) This is an appropriate model for ride-hailing systems (Banerjee et al. 2016, Braverman et al. 2016, Waserhole and Jost 2016), where supply units are typically long-lived. For each location i there are some *compatible supply locations* that are close enough, from where the platform can assign supply units to serve customers at i. Customers arrive stochastically with different origin-destination pairs. Each time a customer arrives, the platform makes an assignment decision from a compatible supply location based on the current spatial distribution of available supplies.

The platform's goal is to maintain adequate supply in all neighborhoods and hence meet as much demand as possible, therefore we adopt the (global) proportion of dropped demands as a measure of efficiency. For the formal description of our model, see Section 2.

To study state-dependent spatial rebalancing of supply while keeping the size of the state space manageable, we make a key simplification – we assume that pickup and service of customers are both instantaneous. This allows us to get away from the complexity of tracking the positions of in transit supply units, while retaining the essence of our focal challenge, that of ensuring that all neighborhoods have supplies at (almost) all times. To obtain tight characterizations, we further consider the asymptotic regime where the number of supply units in the system K

goes to infinity. It's worth noting that the large supply regime (demand arrival rates stay fixed while $K \to \infty$) and the large market regime (demand arrival rate scales with K as $K \to \infty$) are equivalent in our model. The reason is that we assume instantaneous relocation of assigned supply units, hence the large market regime is simply a speed-up of the infinite supply regime. In terms of practical implications for a real setting with transit times, the interpretation of our parameter K is the number of *free* cars in the system at any time. Realistic simulations reveal that our theoretical findings strikingly retain their power even when the above two assumptions are relaxed, i.e., in a setting with transit times, and where only a small number of cars in the system are free at any time since almost all cars are en route for pickup or dropoff (Section 6).

A main assumption in our model is the complete resource pooling (CRP) condition. CRP is a standard assumption in the heavy traffic analysis of queueing systems (see, e.g., Harrison and López 1999, Dai and Lin 2008, Shi et al. 2015). It can be interpreted as requiring enough overlapping in the processing ability of servers so that they form a "pooled server". For the model considered in this paper, the CRP condition is closely related to the condition in Hall's marriage theorem in bipartite matching theory. Violation of Hall's condition would force a positive fraction of demand to be dropped as $K \to \infty$. In a real world ride hailing setting, the demand arrival rates are endogenously determined by system manager's pricing decisions (though we do not incorporate this aspect in the present paper; our model "turns off" pricing and repositioning in the interest of tractability). Intuitively, if CRP condition does not hold for base prices, the platform may use spatially varying prices and repositioning as in Bimpikis et al. (2016), Banerjee et al. (2016) to make it true,[1] and then use our assignment control to manage the unpredictable fluctuations in demand, allowing almost all demand (given the prices) to be fulfilled.

Analogy with classic closed queueing network scheduling problem. Using the terminology of classic queueing theory, the K supply units are "jobs", each demand location is a "server", each supply location is a "buffer", inter-arrival times of customers with origin i are "service times" at server i. The distribution of customers' destinations given origin captures

[1]When we presented our work to them, data scientists at the largest ride-hailing companies proactively suggested the viewpoint that their pricing can ensure our CRP condition, with the possible exception of certain countries where pricing is a less acceptable lever (e.g., China and India).

“routing probabilities”. “Servers” are flexible (i.e. can serve multiple queues), and assignment is equivalent to “scheduling”.

1.1 Main Contributions

We obtain state-dependent control policies that effectively manage the spatial distribution of supply, leading to a fraction of demand dropped that decays exponentially fast in K. This motivates us to perform a thorough *large deviations analysis* which yields surprisingly elegant results. As a function of system primitives, we derive a large deviation rate-optimal assignment policy that minimizes demand dropping (maximizes throughput). Our optimal policy is strikingly simple and its parameters depend in a natural way on demand arrival rates. Our contribution is threefold:

1. **A family of simple policies, and an exponent-optimal policy.** We propose a family of state-dependent assignment policies called Scaled MaxWeight (SMW) policies, and prove that all of them guarantee exponential decay of demand-drop probability under the CRP condition. The proof is based on a family of novel policy-dependent Lyapunov functions which are used to analyze a multi-dimensional variational problem. An SMW policy is parameterized by a vector of scaling factors, one for each location; each demand is served by assigning a supply from the compatible location with the largest scaled number of cars. We obtain an explicit specification for the optimal scaling factors based on location pickup compatibilities and demand arrival rates. Further, we obtain the surprising finding that the optimal SMW policy is, in fact, *exponent-optimal* among all state-dependent policies (Theorem 1). SMW policies are simple, explicit and promising for practical applications (Section 6 demonstrates stellar performance in a realistic simulation environment).
2. **The value of state-dependent control.** We show that no state-independent assignment policy can achieve exponential decay of demand dropping (Proposition 2), which demonstrates the value of state-dependent control — even the naive unscaled (“vanilla”) MaxWeight assignment policy requiring no knowledge of demand arrival rates achieves exponential decay.
3. **Qualitative insights.** We establish the *critical subset* property of the SMW policies: for each SMW policy, there exist subsets of demand locations that are most vulnerable to the

depletion of supply in compatible locations, hence leading to demand dropping. The optimal SMW policy simultaneously "protects" all critical subsets maximally by maintaining high supply levels near structurally under-supplied locations.

One key difficulty in the analysis is the necessity to deal with a multi-dimensional system even in the limit. In many existing works that seek to minimize the workload/holding costs of a queueing system, asymptotic optimality of a certain policy relies on "collapse" of the system state to a lower dimensional space in the heavy traffic limit. In contrast, in our setting, the limit system remains n-dimensional, where n is the number of locations.

1.2 Literature Review

MaxWeight scheduling. MaxWeight is a simple scheduling policy in constrained queueing networks which (roughly speaking) chooses the feasible control decision that serves the queues with largest total weight, at each time. Examples of weight include queue lengths, scaled queue lengths, etc. MaxWeight scheduling has been shown to exhibit good performance in various settings (e.g., Tassiulas and Ephremides 1992, Dai and Lin 2005, Stolyar 2004, Dai and Lin 2008, Eryilmaz and Srikant 2012, Maguluri and Srikant 2016), including by Shi et al. (2015) who study an open one-hop network version of our setting. In contrast, we find that MaxWeight is suboptimal in our closed network setting.

Large deviations in queueing systems. There is a large literature on characterizing the probability of building up long queues in *open* queueing networks, including controlled (see, e.g., Stolyar and Ramanan 2001, Stolyar 2003) and uncontrolled (see, e.g., Majewski and Ramanan 2008, Blanchet 2013) networks. The work closest to ours is that of Venkataramanan and Lin (2013), who established the relationship between Lyapunov functions and buffer overflow probability for open queueing networks. The key difficulty in extending the Lyapunov approach to closed queueing networks is the lack of a natural reference state where the Lyapunov function equals to 0 (in an open queueing network the reference state is simply $\mathbf{0}$). It turns out that as we optimize the MaxWeight parameters we are also solving for the best reference state.

Shared transportation systems. Ozkan and Ward (2016) studied revenue-maximizing state-independent assignment control by solving a minimum cost flow problem in the fluid limit.

Braverman et al. (2016) modeled the system by a closed queueing network and derived the optimal static routing policy that sends empty vehicles to under-supplied locations. Banerjee et al. (2016) adopted the Gordon-Newell closed queueing network model and considered static pricing/repositioning/matching policies that maximizes throughput, welfare or revenue. In contrast to our work, which studies state-dependent control, these works consider static control that completely relies on system parameters. In terms of convergence rate to the fluid-based solution, Ozkan and Ward (2016) did not study the convergence rate of their policy, Braverman et al. (2016) observed from simulation an $O(1/\sqrt{K})$ convergence rate as the number of supply units in the closed system K goes to infinity,[2] while Banerjee et al. (2016) showed finite system bounds with an $O(1/K)$ convergence rate as $K \to \infty$ in the absence of service times and an $O(1/\sqrt{K})$ convergence rate with service times. Under the CRP condition, we obtain exponential decay in K of the demand drop probability, and further obtain the optimal exponent.

Other recent works study pricing aspects of ride-hailing; see, e.g., Adelman (2007), Bimpikis et al. (2016), Waserhole and Jost (2016), Cachon et al. (2017), Hall et al. (2015).

Online stochastic bipartite matching. There is a related stream of research on online stochastic bipartite matching, see, e.g., Caldentey et al. (2009), Adan and Weiss (2012), Buỳić and Meyn (2015), Mairesse and Moyal (2016). Different types of supplies and demands arrive over time, and the system manager matches supplies with demands of compatible types using a specific matching policy, and then discharges the matched pairs from the system. Our work is different in that we study a *closed* system where supply units never enter or leave the system. Moreover, this literature focuses on the stability and other properties under a given policy instead of looking for the optimal control (except Buỳić and Meyn 2015).

Other related work. Jordan and Graves (1995), Désir et al. (2016), Shi et al. (2015) and others study how process flexibility can facilitate improved performance, analogous to our use of dispatch control to improve demand fulfillment. Along similar lines, network revenue management is a classical dynamic resource allocation problem, see, e.g., Gallego and Van Ryzin (1994), Talluri and Van Ryzin (2006), and recent works, e.g., Jasin and Kumar (2012), Bumpen-

[2] In the setting of Braverman et al. (2016), the drop probability can remain positive even as K grows, in contrast with our setting where the drop probability can always be sent to 0 because of our CRP condition under which the flows in the network can potentially be balanced. The comparison of convergence rates is most meaningful if we restrict attention to instances in their setting where the drop probability goes to zero as K grows.

santi and Wang (2018). Different types of demands arrive over time, and a centralized decision is made at each arrival. Again, each of these settings is "open" in that each service token or supply unit can be used only once, in contrast to our setting.

1.3 Organization of the paper

The remainder of our paper is organized as follows. In Section 2 we introduce the basic notation and formally describe our model together with the performance metric. In Section 3 we introduce the family of Scaled MaxWeight policies. In Section 4 we present our main theoretical result, i.e., that there is an exponent optimal SMW policy for any set of primitives. In Section 5 we prove the exponent optimality of SMW policies. In Section 6 we describe our simulation study of SMW policies using NYC yellow cab data. We conclude in Section 7.

2 Model and Preliminaries

2.1 Notation

We use $\mathbf{e}_i$ to denote the i-th unit vector, and $\mathbf{1}$ the all-1 vector, both of which are n-dimensional. For index set $A \in \{1, \cdots, n\}$, define $\mathbf{1}_A \triangleq \sum_{i \in A} \mathbf{e}_i$. For a set Ω in Euclidean space $\mathbb{R}^n$, denote its relative interior by relint(Ω). We use $B(\mathbf{x}, \epsilon)$ to denote a ball centered at $\mathbf{x} \in \mathbb{R}^n$ with radius $\epsilon > 0$. For event C, we define the indicator random variable $\mathbb{I}\{C\}$ to equal 1 when C is true, else 0. All vectors are column vectors if not specified otherwise.

2.2 Basic Setting

Underlying Model and Simplifications: We model the shared transportation system as a finite-state Markov process, comprising of a fixed number of identical *supply units* circulating among n *nodes* (i.e., a given partition of a city into neighborhoods). Customers (i.e., prospective passengers) arrive at each node j with desired destination k according to independent Poisson processes with rate $\hat{\phi}_{jk}$. To serve an arriving customer, the platform must immediately assign a supply unit from a "neighboring" node of j (i.e., one among a set of adjacent neighborhoods, defined formally below), and subsequently, after serving the customer, the supply unit becomes available at the destination node k. If, however, the customer is not assigned a supply unit, for example because none is available at any neighboring nodes of j, then we experience a *demand drop*, wherein the customer leaves the system without being served. Customers do not wait. The aim of the platform is to assign supplies so as to minimize the fraction of demand dropped. Intuitively, to achieve this objective, the platform should ensure that it maintains adequate supply in (or near) all neighborhoods, i.e., it needs to manage the spatial distribution of supply.

To study the design of assignment rules in the above model, we make some simplifications. First, we assume that pickup and service are instantaneous. This allows us to reformulate the above model as a *discrete-time* Markov chain (the so-called *jump chain* of the continuous-time process), where in each time slot $t \in \mathbb{N}$, with probability proportional to $\hat{\phi}_{jk}$, exactly one customer arrives to the system at node j with desired destination k. The customer is then served by an assigned supply unit from a neighboring station of j, which then becomes available at node k at the beginning of time slot $t+1$. This simplification removes the infinite dimensional state required for tracking the positions of all in-transit supply units, while still retaining the complex supply externalities between stations, which is the hallmark of ride-hailing systems. Unfortunately, however, even after this simplification, the setting is still not amenable to a performance characterization of complex assignment policies. This motivates us to study the performance of assignment policies as the number of supply units K grow to infinity, while fixing all other parameters. Intuitively, K is interpreted as the number of *free* cars in the system (typically only a small fraction of the total number of cars in a real system; see Section 6). Finally, we assume that supply units do not relocate except to pick up or drop off a customer.

Formal System Definition: We define $\phi \in \mathbb{R}^{n \times n}$ to be the arrival rate matrix with a row for each origin and a column for each destination, normalized[3] such that $\mathbf{1}^T \phi \mathbf{1} = 1$. We denote the k-th column (i.e., the arrival rates at different origins of customers with destination k) as $\phi_{(k)}$, and the transpose of the j'-th row of the arrival matrix (i.e., the arrival rates of customers with origin j' and different destination nodes) as $\phi_{j'}$. Thus, the probability a customer arrives at node j' is $\mathbf{1}^T \phi_{j'}$, and, assuming all customers are matched, the rate of supply units arriving

[3]This can always be achieved by appropriately re-scaling the arrival rates $\{\hat{\phi}_{jk}\}$, which produces an equivalent setting since pickups and dropoffs are instantaneous.

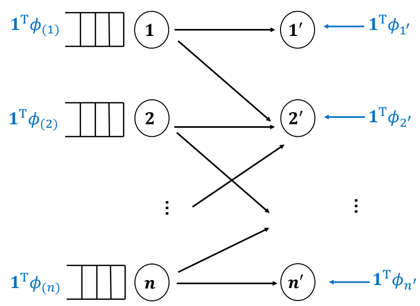

Figure 1: The bipartite (assignment) compatibility graph: On the left are supply nodes $i \in V_S$, and on the right are demand nodes $j' \in V_D$. The edges entering a demand node j' encode compatible (i.e., nearby) nodes that can supply node j'. Customers arrive to j' with different destinations at rates $\phi_{j'}$; the total probability of a customer arrival at j' at each time is thus $\mathbf{1}^T\phi_{j'}$. Similarly, assuming no demand is dropped, the total probability a supply unit arrives at i at each time is $\mathbf{1}^T\phi_{(i)}$.

at node k is $\mathbf{1}^T\phi_{(k)}$.

As mentioned above, we consider a sequence of systems parameterized by the number of supplies K. For the K-th system, its state at any time $t \in \mathbb{N}$ is given by $\mathbf{X}^K[t]$, a vector that tracks the number of supplies at each location in time slot t. The state space of the K-th system is thus given by $\Omega_K \triangleq \{\mathbf{x} \in \mathbb{R}^n | \mathbf{1}^T\mathbf{x} = K\} \cap \mathbb{N}^n$. Note that the normalized state $\frac{1}{K}\mathbf{X}^K[t]$ lies in the n-probability simplex $\Omega = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \geq 0, \mathbf{1}^T\mathbf{x} = 1\}$. Henceforth, we drop explicit dependence on t when it is clear from the context.

2.3 Assignment Policies and System Dynamics

The assignment problem in ride hailing is challenging because most arriving customers have a maximum tolerance, say 7 minutes, for the pickup time or ETA (i.e., expected time of arrival) of a matched supply unit. Thus when a customer arrives at a node j, then any supply unit located at a node which is within 7 minutes of j is a feasible match, while other vehicles which are further away are infeasible. Motivated by this, we use a compatibility graph to capture (binary) assignment compatibility between nodes in a stylized manner.

Compatibility Graph: For pedagogical reasons, we move to a setup where we distinguish formally between demand locations V_D where customers arrive and supply locations V_S where supply units wait (and where customers are dropped off). We add a prime symbol to the indices of nodes in V_D to distinguish between the two. We encode the compatibility graph as a bipartite graph $G(V_S \cup V_D, E)$, wherein each station $i \in V$ is replicated as a supply node $i \in V_S$ and a demand node $i' \in V_D$. (We exclude demand locations $\{i' : \mathbf{1}^T \phi_{i'} = 0\}$ with zero demand arrival rate from V_D.) An edge $(i, j') \in E$ represents a compatible pair of supply and demand nodes, i.e., supplies stationed at i can serve demand arriving at j'. See Figure 1 for an illustration. We denote the neighborhood of a supply node $i \in V_S$ (resp. demand node $j' \in V_D$) in G as $\partial(i) \subseteq V_D$ (resp. $\partial(j') \subseteq V_S$); thus, for a supply node i, its compatible demand nodes are given by $\partial(i) = \{j' \in V_D | (i, j') \in E\}$, and similarly for each demand node. Moreover, for any set of supply nodes $A \subseteq V_S$, we also use $\partial(A)$ to denote its demand neighborhood (and vice versa).

We use the term *network* to refer to a given set of primitives: an assignment compatibility graph G and demand arrival rates ϕ. We make a mild assumption on arrival rates ϕ.

Assumption 1. (Non-triviality). *There exists an origin-destination pair $j' \in V_D$ and $k \in V_S$ such that $k \notin \partial(j')$ and $\phi_{j'k} > 0$, i.e., the destination k for these customers is not a supply location compatible with their origin j'.*

Assumption 1 is made to ensure that the assignment control problem at hand is non-trivial. If it is violated, and if the system starts with at least one supply unit in each location, then we can "reserve" a supply unit for each demand origin location $j' \in V_D$, and each such car will never leave the corresponding neighborhood $\partial(j')$, ensuring that no demand is ever dropped.

Assignment policies: Given the above setting, the problem we want to study is how to design assignment policies which minimize the probability of dropping demand. For fixed K, this problem can be formulated as an average cost Markov decision process on a finite state space, and is thus known to admit a stationary optimal policy (i.e., where the assignment rule at time t only depends on the system state $\mathbf{X}^K[t]$; see Proposition 5.1.3 in Bertsekas 1995). Let $\mathcal{U}^K$ be the set of stationary policies for the K-th system.

For each $t \in \mathbb{N}$, $j' \in V_D$, $k \in V_S$, an assignment policy $U \in \mathcal{U}$ consists of a sequence of mappings $\left(U^K \in \mathcal{U}^K\right)_{K=1}^{\infty}$, which map the current queue-length vector to $U^K[\mathbf{X}^K[t]](j', k) \in \partial(j') \cup \{\emptyset\}$. Here $U^K[\mathbf{X}^K[t]](j', k) = i$ means given the current state $\mathbf{X}^K[t]$, we assign a supply unit from $i \in \partial(j')$ to fulfill demand at j' that goes to k, and $U^K[\mathbf{X}^K[t]](j', k) = \emptyset$ means that the platform does not assign supplies to type (j', k) demands and hence any such demand is dropped. When $\mathbf{X}_i^K[t] = 0$ for all $i \in \partial(j')$, this forces $U^K[\mathbf{X}^K[t]](j', k) = \emptyset$ since there is no supply at locations compatible to j'. For simplicity of notation, we refer to the policies by U instead of U^K.

System Evolution: We can now formally define the evolution of the Markov chain we want to study. At the beginning of time slot t, the state of the system is $\mathbf{X}^K[t-1]$; note that this incorporates the state change due to serving the demand in time slot $t-1$. Now suppose the platform uses an assignment policy U, and in time slot t, a customer arrives at origin node $o[t]$ with destination $d[t]$ (sampled with joint distribution as specified by arrival matrix ϕ). If $U^K[\mathbf{X}^K[t]](o[t], d[t]) \neq \emptyset$, then a supply unit from $U^K[\mathbf{X}^K[t]](o[t], d[t])$ will pick up the demand and relocate to $d[t]$ instantly. Let $S[t] \triangleq U^K[\mathbf{X}^K[t]](o[t], d[t])$ be the chosen supply node (potentially $\emptyset$). Formally, we have

$$\mathbf{X}^K[t] \triangleq \begin{cases} \mathbf{X}^K[t-1] - \mathbf{e}_{S[t]} + \mathbf{e}_{d[t]} & \text{if } S[t] \in V_S\,, \\ \mathbf{X}^K[t-1] & \text{if } S[t] = \emptyset\,. \end{cases}$$

2.4 Performance Measure

The platform's goal is to find an assignment policy that drops as few demands as possible in steady state. A natural performance measure is the *long-run average demand-drop probability.* Formally, for $U \in \mathcal{U}$ we define

$$\mathbb{P}_{\mathrm{o}}^{K,U} \triangleq \min_{\mathbf{X}^{K,U}[0] \in \Omega_K} \mathbb{E}\left(\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \mathbb{I}\left\{ U^K[\mathbf{X}^{K,U}[t]](o[t], d[t]) = \emptyset \right\} \right), \tag{1}$$

$$\mathbb{P}_{\mathrm{p}}^{K,U} \triangleq \max_{\mathbf{X}^{K,U}[0] \in \Omega_K} \mathbb{E}\left(\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \mathbb{I}\left\{ U^K[\mathbf{X}^{K,U}[t]](o[t], d[t]) = \emptyset \right\} \right). \tag{2}$$

Here (1) is an *optimistic* (subscript "o" for optimistic) performance measure (which underestimates demand-drop probability), whereas (2) is a *pessimistic* (subscript "p" for pessimistic) performance measure (which overestimates demand-drop probability). Since $U \in \mathcal{U}$ is a stationary policy, the limits in (1) and (2) exist. Note that $\mathbb{P}_{\mathrm{o}}^{K,U} \leq \mathbb{P}_{\mathrm{p}}^{K,U}$. We will establish the exponent optimality of our policy by showing that its pessimistic measure decays as fast with

K as any policy's optimistic measure can possibly decay.

The exact values of (1) and (2) for fixed K are challenging to study. To this end, the main performance measures of interest in this work are the decay rates of $\mathbb{P}_{\mathrm{o}}^{K,U}$ and $\mathbb{P}_{\mathrm{p}}^{K,U}$ as $K \to \infty$:

$$\gamma_{\mathrm{o}}(U) \triangleq -\liminf_{K\to\infty} \frac{1}{K} \log \mathbb{P}_{\mathrm{o}}^{K,U}\,, \tag{3}$$

$$\gamma_{\mathrm{p}}(U) \triangleq -\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}_{\mathrm{p}}^{K,U}\,. \tag{4}$$

For brevity, we henceforth refer to these as the *demand-drop exponents*. Note that $\gamma_{\mathrm{o}}(U) \geq \gamma_{\mathrm{p}}(U)$. The definition (3) uses $\liminf$ so that we can state a strong converse result by upper bounding $\sup_{U\in\mathcal{U}} \gamma_{\mathrm{o}}(U)$, since no policy can achieve a larger demand-drop exponent. Similarly, the definition (4) uses $\limsup$ so that we can state a strong achievability result (for our proposed policies the limit will exist; when the limit exists we write $\gamma(U) \triangleq \gamma_{\mathrm{o}}(U) = \gamma_{\mathrm{p}}(U)$).

2.5 Sample Path Large Deviation Principle

Our result relies on classical large deviation theory, which we briefly introduce in this subsection.

For each fixed $K \in \mathbb{N}_+$ and $T \in (0,\infty)$, define a *scaled sample path* of accumulated demand arrivals $\bar{\mathbf{A}}^K[\cdot] \in (L^\infty[0,T])^{n^2}$ as follows. For $t = 1/K, 2/K, \ldots, \lceil KT \rceil/K$, let

$$\bar{\mathbf{A}}^K_{j'k}[t] \triangleq \frac{1}{K}\mathbf{A}^K_{j'k}[Kt]\,, \quad \text{for } \mathbf{A}^K_{j'k}[t'] \triangleq \sum_{\tau=1}^{t'} \mathbb{I}\{o[\tau]=j', d[\tau]=k\}\,. \tag{5}$$

$\bar{\mathbf{A}}^K[t]$ is defined by linear interpolation for other t's. Let μ_K be the law of $\bar{\mathbf{A}}^K[\cdot]$ in $(L^\infty[0,T])^{n^2}$. For all $\mathbf{f} \in \mathbb{R}^{n\times n}$, let

$$\Lambda^*(\mathbf{f}) \triangleq \begin{cases} D_{\mathrm{KL}}(\mathbf{f}||\phi) & \text{if } \mathbf{f} \geq 0, \mathbf{1}^T\mathbf{f}\mathbf{1} = 1\,, \\ \infty & \text{otherwise}\,. \end{cases} \tag{6}$$

Here $D_{\mathrm{KL}}(\mathbf{f}||\phi)$ is Kullback-Leibler divergence between $\mathbf{f}$ and ϕ defined as:

$$D_{\mathrm{KL}}(\mathbf{f}||\phi) = \sum_{j'\in V_D} \sum_{k\in V_S} f_{j'k} \log \frac{f_{j'k}}{\phi_{j'k}}\,.$$

For any set Γ, let $\bar{\Gamma}$ be its closure, and Γ^o be its interior. Below is the sample path large deviation principle (a.k.a. Mogulskii's Theorem, see Dembo and Zeitouni 1998):

Fact 1. *For measures $\{\mu_K\}$ defined above, and any arbitrary measurable set $\Gamma \subseteq (L^\infty[0,T])^{n^2}$, we have*

$$-\inf_{\bar{\mathbf{A}} \in \Gamma^o} I_T(\bar{\mathbf{A}}) \le \liminf_{K\to\infty} \frac{1}{K} \log \mu_K(\Gamma) \le \limsup_{K\to\infty} \frac{1}{K} \log \mu_K(\Gamma) \le -\inf_{\bar{\mathbf{A}} \in \bar{\Gamma}} I_T(\bar{\mathbf{A}}), \tag{7}$$

where the rate function[4] *is:*

$$I_T(\bar{\mathbf{A}}) \triangleq \begin{cases} \int_0^T \Lambda^* \left(\frac{d}{dt} \bar{\mathbf{A}}(t) \right) dt & \text{if } \bar{\mathbf{A}}(\cdot) \in \mathrm{AC}[0,T], \bar{\mathbf{A}}(0) = \mathbf{0}, \\ \infty & \text{otherwise}. \end{cases} \tag{8}$$

Here $\mathrm{AC}[0,T]$ *is the space of absolutely continuous functions on* $[0,T]$.

Informally, this fact says the following. (Suppose the leftmost term and rightmost term in (7) are equal.) The probability exponent (with respect to K) for the unlikely event Γ is equal to the exponent for the most likely fluid sample path (a limit of scaled sample paths, see Section 5.1) of demand $\bar{\mathbf{A}}$ such that the event occurs. The exponent for $\bar{\mathbf{A}}$ is the *time integral of the exponent for its time derivative.*

In our case, Γ is the demand-drop event. We first derive a converse bound on the exponent by constructing a limiting sample path that always leads to demand drop regardless of the policy. Then we analyze the performance of our proposed SMW policies by looking at the limiting sample paths that lead to demand drop under those policies.

3 Scaled MaxWeight Policies

We now introduce the family of scaled MaxWeight (SMW) policies. The traditional MaxWeight policy (hereafter referred to as vanilla MaxWeight) is a dynamic scheduling rule that allocates the service capacity to the queue(s) with largest "weight" (where weight can be any relevant parameter such as queue-length, sum of queue-lengths, head-of-the-line waiting time, etc.). In our setting, supply units are like jobs and demand is like service tokens, and vanilla MaxWeight would correspond to assigning from the compatible location with most supply units (with appropriate tie-breaking rules).

The popularity of MaxWeight scheduling stems from the fact that it is known to be optimal

[4]Since absolutely continuous functions are differentiable almost everywhere, the rate function is well-defined.

for different metrics in various problem settings (e.g., refer Stolyar 2003, 2004, Shi et al. 2015, Maguluri and Srikant 2016). However, in our setting, vanilla MaxWeight is suboptimal (and further, we will show that it does not achieve the optimal exponent). The suboptimality of vanilla MaxWeight can be seen from the following simple example.

Example 1. *Consider a closed network with two nodes* $\{1,2\}$*, compatibility graph* $G = (V_S \cup V_D, E) = (\{1,2\} \cup \{1', 2'\}, \{11', 12', 22'\})$ *and demand arrival rates* $\phi_{1'1} = 3/8, \phi_{1'2} = 1/8, \phi_{2'1} = \phi_{2'2} = 1/4$*, as shown in Figure 2. Suppose at time* t *we have* $\mathbf{X}_1[t] > \mathbf{X}_2[t]$ *and a demand arrives at node* $2'$*.*

Under the vanilla MaxWeight policy, we would assign from node 1 *since there are more supply units there. However, we claim that vanilla MaxWeight is dominated (in terms of minimizing demand dropping probability) by another policy where one always assigns from node* 2 *to serve demand at* $2'$ *as long as* $\mathbf{X}_2 > 0$*. We call this policy the priority policy. To see this intuitively, note that under both policies demand at node* $2'$ *will never be dropped, hence demand dropping happens if and only if supply at node* 1 *is depleted. The priority policy tries to keep all the servers at node* 1 *while vanilla MaxWeight tries to equal the number of servers on both nodes, hence the priority policy drops less demand. In fact, as we will show formally later, the exponent of demand dropping under the priority policy is twice as large as the exponent under vanilla MaxWeight. (Here the priority policy is optimal, and this is true for any* ϕ*. In general, the optimal policy depends on both* G *and* ϕ*, and vanilla MaxWeight is generically suboptimal.)*

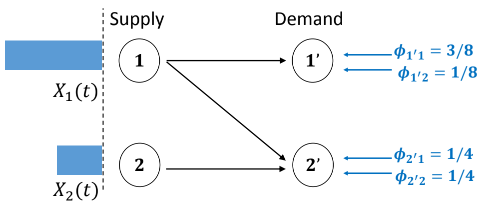

Figure 2: An example of the sub-optimality of vanilla MaxWeight policy.

To deal with this issue, we generalize vanilla MaxWeight by attaching a positive scaling parameter α_i to each queue $i \in V_S$, and assign from the compatible queue with largest *scaled* queue length $\mathbf{X}_i/\alpha_i$. Without loss of generality, we normalize α s.t. $\mathbf{1}^T\alpha = 1$, or equivalently,

$\alpha \in \text{relint}(\Omega)$.

We call this family of policies *Scaled MaxWeight (*SMW*) policies*, and use SMW(α) to denote SMW with parameter α. Going back to Example 1, we can approximate the priority policy by attaching a much larger scaling parameter at node 1 than at node 2.

The formal definition of SMW is as follows.

Definition 1 (Scaled MaxWeight SMW(α))**.** *Given system state* $\mathbf{X}[t-1]$ *at the start of* t*-th period and for demand arriving at* $o[t]$*,* SMW*(α) assigns from*

$$\text{argmax}_{i \in \partial(o[t])} \frac{\mathbf{X}_i[t-1]}{\alpha_i}$$

if $\max_{i \in \partial(o[t])} \frac{\mathbf{X}_i[t-1]}{\alpha_i} > 0$*; otherwise the demand is dropped. If there are ties when determining the argmax, assign from the location with highest index.*

Remark 1. *Since CRP holds, nearly all demands are fulfilled as* $K \to \infty$ *regardless of their destinations. Since the destination cannot be controlled, it turns out that the (asymptotic) performance of SMW policies is not hurt by the fact that they ignore the customers' destinations.*

As may be expected, SMW policies tend to equalize the scaled queue lengths if CRP holds. The following fact is formalized in Section 5, en route to proving our main result.

Remark 2 (Resting point of state under SMW(α))**.** *If Assumption 2 below holds, under* SMW(α) *policy the normalized system state* $\mathbf{X}^K/K$ *in steady state concentrates as* $K \to \infty$ *at* α*, where all the scaled queue lengths are equal to* 1*.*

4 Main Result

In this section we present our main result, namely, that for any network such that CRP holds, there exists an α such that SMW(α) is exponent optimal. Along the way we establish that all SMW policies yield exponential decay of demand dropping. In contrast, we show that no state independent policy can produce exponential decay (Section 4.1).

Complete Resource Pooling Condition. The following is the main assumption of this paper.

Assumption 2. *We assume that for any $J \subsetneq V_D$ where $J \neq \emptyset$,*

$$\sum_{i \in \partial(J)} \mathbf{1}^T \phi_{(i)} > \sum_{j' \in J} \mathbf{1}^T \phi_{j'} \,. \tag{9}$$

The intuition behind this assumption is simple: it assumes the system is "balanceable" in that for each subset $J \subsetneq V_D$ of demand locations we have enough supply at neighboring locations to meet the demand. Assumption 2 is equivalent to a strict version of the condition in Hall's marriage theorem. It is also closely related to the Complete Resource Pooling (CRP) condition in queueing (Harrison and López 1999, Stolyar 2004, Ata and Kumar 2005, Gurvich and Whitt 2009). This assumption marks the limit of assignment policies — no assignment policy can achieve exponentially decaying demand drop probability when Assumption 2 is violated.

Proposition 1. *For any G and ϕ's such that Assumption 2 is violated, it holds that for any policy U, the demand dropping probability does not decay exponentially,*[5] *i.e., $\gamma_{\mathrm{o}}(U) = \gamma_{\mathrm{p}}(U) = 0$ where $\gamma_{\mathrm{o}}(U)$ and $\gamma_{\mathrm{p}}(U)$ are defined in* (3) *and* (4)*.*

In other words, if Assumption 2 is violated, this means the system has significant spatial imbalance of demand and demand dropping is unavoidable (unless stronger forms of control like pricing or repositioning are deployed to restore spatial balance). The proof of Proposition 1 is in Appendix A.

Rate Optimality of Scaled MaxWeight. In the fluid limit, it is easy to find an assignment rule such that no demand is dropped (see, e.g., Banerjee et al. 2016). Our goal here is to approach zero demand dropping as fast as possible as the number of supply units K grows. Define

$$\mathcal{J} \triangleq \left\{ J \subsetneq V_D : \sum_{j' \in J} \sum_{k \notin \partial(J)} \phi_{j'k} > 0 \right\}. \tag{10}$$

We have $\mathcal{J} \neq \emptyset$ by Assumption 1. The following is our main result.

Theorem 1 (Main Result). *For any network (G, ϕ) satisfying Assumptions 1 and 2, we have:*

1. ***Achievability**: For any $\alpha \in \mathrm{relint}(\Omega)$, $\mathrm{SMW}(\alpha)$ achieves exponential decay of the demand*

[5] In fact, if the inequality (9) in Assumption 2 is strictly *reversed* for some $J \subsetneq V_D$, then we have a demand dropping probability which is at least $\epsilon > 0$ for all K, where $\epsilon = \sum_{j' \in J} \mathbf{1}^T \phi_{j'} - \sum_{i \in \partial(J)} \mathbf{1}^T \phi_{(i)}$.

dropping probability with exponent[6,7]

$$\gamma(\alpha) = \min_{J \in \mathcal{J}} B_J \log \left(\frac{\lambda_J}{\mu_J} \right) > 0 \,, \tag{11}$$

$$\text{where} \quad B_J \triangleq \mathbf{1}_{\partial(J)}^{\mathrm{T}} \alpha \,, \qquad \lambda_J \triangleq \sum_{j' \notin J} \sum_{k \in \partial(J)} \phi_{j'k} \,, \quad \text{and} \quad \mu_J \triangleq \sum_{j' \in J} \sum_{k \notin \partial(J)} \phi_{j'k} \,.$$

2. ***Converse:*** *Under any policy U, it must be that*

$$\gamma_{\mathrm{p}}(U) \le \gamma_{\mathrm{o}}(U) \le \gamma^* \,, \qquad \text{where} \quad \gamma^* = \sup_{\alpha \in \mathrm{relint}(\Omega)} \gamma(\alpha) \,. \tag{12}$$

Thus, there is an SMW *policy that achieves an exponent arbitrarily close to the optimal one.*

We will prove Theorem 1 in Section 5. The first part of the theorem states that for any SMW policy with α in the relative interior of Ω, the policy achieves an explicitly specified positive demand drop exponent $\gamma(\alpha)$, i.e., the demand dropping probability decays as $e^{-(\gamma(\alpha)-o(1))K}$ as $K \to \infty$. The second part of the theorem provides a universal upper bound γ^* on the exponent that any policy can achieve, i.e., for any assignment policy U, the demand dropping probability is at least $e^{-(\gamma^*+o(1))K}$. Crucially, γ^* is identical to the supremum over α of $\gamma(\alpha)$. In other words, *there is an (almost) exponent optimal* SMW *policy*, and moreover, the scaling parameters for this policy can be obtained as the solution to the explicit problem: $\text{maximize}_{\alpha \in \mathrm{relint}(\Omega)} \gamma(\alpha)$.

We note that this result is qualitatively different from the numerous results showing optimality of (vanilla) maximum weight matching in various open queuing network settings. Here, our objective is a natural objective that is symmetric in all the queues. Our result says that there is an optimal *scaled* maximum weight policy, that is *not* symmetric; rather, it is asymmetric via specific scaling factors that optimally account for the primitives of the network.

The following remark provides some intuition regarding the expression for $\gamma(\alpha)$.

Remark 3 (Intuition for $\gamma(\alpha)$)**.** *Consider the expression for $\gamma(\alpha)$ in* (11)*. It is a minimum of a "robustness" term for each subset $J \in \mathcal{J}$ of demand locations. For subset J, the robustness of* SMW(α)*'s ability to serve demand arising in J is the product of two terms (see Figure 3 for an*

[6] We show that for SMW policies, the lim inf in (3) and lim sup in (4) are equal, i.e., $\gamma_{\mathrm{o}}(\alpha) = \gamma_{\mathrm{p}}(\alpha)$. (We use α to represent the policy SMW(α) in the argument of the γs.)

[7] Note that the argument of the logarithm has a strictly larger numerator than denominator for every $J \subsetneq V_D$ since Assumption 2 holds, implying that $\gamma(\alpha)$ is the minimum of finitely many positive numbers, and hence is positive.

illustration of the quantities involved):

- *Robustness arising from α: At the resting point α (see Remark 2) of* SMW(α)*, the supply at neighboring locations is $B_J = \mathbf{1}^T_{\partial(J)}\alpha$, and the larger that is, the more unlikely it is that the subset will be deprived of supply.*
- *Robustness arising from excess of supply over demand: The logarithmic term* $\log(\lambda_J/\mu_J)$ *captures how vulnerable that subset is to being drained of supply. Observe that λ_J is the maximum average rate at which supply can come in to $\partial(J)$ from $V_D\backslash J$, whereas μ_J is the minimum average rate at which supply goes from J to $V_S\backslash\partial(J)$ if all demand in J is served (even if demand in $V_D\backslash J$ is served from other locations). The larger the ratio, the more oversupplied and hence robust J is.*

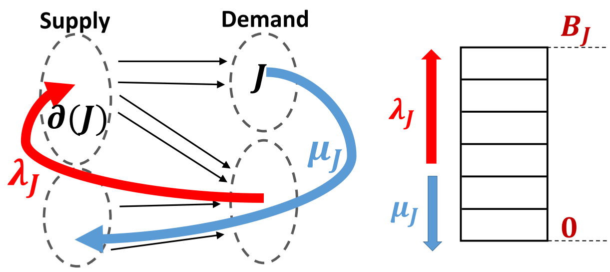

Figure 3: An illustration of the terms B_J, λ_J, and μ_J in Theorem 1.

Remarkably, the expression for robustness of subset J is as large (i.e., as good) as the demand drop exponent for J would be with starting state α under a "protect-J" policy which exclusively protects J at the expense of all other locations. *(Similar to standard buffer overflow probability calculations, the likelihood of the supply at $\partial(J)$ being depleted by KB_J units under a protect-J policy is $\Theta((\mu_J/\lambda_J)^{-KB_J}) = \Theta(\exp(-KB_J\log(\lambda_J/\mu_J)))$. We then set B_J to the starting scaled supply at $\partial(J)$, i.e., $B_J = \mathbf{1}^T_{\partial(J)}\alpha$, to establish the claim.) Thus, Theorem 1 part 1 says that given the resting state α,* SMW(α) *achieves an exponent such that* it suffers no loss from the need to protecting multiple J's simultaneously. *It is then intuitive that the globally optimal exponent can be achieved via an SMW policy by choosing α suitably.*

Next, we discuss the optimal choice of the resting state α based on Theorem 1.

Remark 4 (Intuition for optimal α)**.** *Informally, consider the special case of a "heavy traffic"* *type setting where there is just one subset J which has a vanishing logarithmic term (because it is only very slightly oversupplied, with λ_J only slightly more than μ_J), whereas each other subset of V_D has a logarithmic term that converges to some positive number. Then the optimal choice of α will satisfy $B_J = \mathbf{1}_{\partial(J)}^T \alpha \to 1$, i.e., all but a fraction of the supply will be at locations that can serve J in the resting state. The intuition is that the random walk for the supply at $\partial(J)$ has only slightly positive drift even if the assignment rule protects it, and hence it is optimal to keep the total supply at these locations at a high resting point, to minimize the likelihood of depletion. We think an optimal policy for such a special case is itself interesting; Theorem 1 goes much beyond to solve the general n-dimensional problem considering* all *subsets simultaneously. The optimal α protects supply that can serve structurally undersupplied locations.*

We establish our result via a *novel Lyapunov analysis for a closed queueing network.* A key technical challenge we face in our closed queueing network setting is that it is a priori unclear what the "best" state is for the system to be in. This is in contrast to open queueing network settings in which the best state is typically the one in which all queues are empty, and the Lyapunov functions considered typically achieve their minimum at this state. We get around this issue via an innovative approach where we define a policy-specific Lyapunov function that achieves its minimum at the resting point of the SMW policy we are analyzing, and use this Lyapunov function to characterize its exponent $\gamma(\alpha)$. Moreover, given the optimal choice of α, our tailored Lyapunov function corresponding to this choice of α helps us establish our converse result. Our analysis is described in Section 5.

4.1 No State-Independent Policy Achieves Exponential Decay

To demonstrate the value of state-dependent control, we show that exponential decay of demand-drop probability is impossible under state-independent policies as studied in Ozkan and Ward (2016) and Banerjee et al. (2016). We first formally define state-independent policies.

Definition 2. *We call a dispatch policy π state independent if it maps each $j' \in V_D$, $k \in V_S$, $t \in \mathbb{N}$ to a distribution $u_{j'k}[t]$ over $\partial(j') \cup \{\emptyset\}$; for a demand with origin j' and destination k that arrives at time t, the platform dispatches from i drawn independently from distribution $u_{j'k}[t]$,* *ignoring the current state* $\bar{\mathbf{X}}[t-1]$ *and the history. If* $i = \emptyset$ *or there is no supply at the dispatch location, the demand is dropped.*

Proposition 2. *Under any state-independent dispatch policy* π*, we have:* $P^{K,\pi} = \Omega\left(\frac{1}{K^2}\right)$*. In particular,* $\gamma_{\rm o}(\pi) = 0$*, where* $\gamma_{\rm o}(\cdot)$ *is the optimistic exponent defined in* (3)*.*

The proof of Proposition 2 is in Appendix A.

5 Analysis of Scaled MaxWeight Policies

In this section, we prove the exponent optimality of SMW policies and derive explicitly the optimal exponent and most likely sample paths leading to demand drop. In Section 5.1, we follow the standard approach and characterize the system behavior in the fluid scale through fluid sample paths and fluid limits. In Section 5.2 we define a novel family of Lyapunov functions. In Section 5.3 we follow the approach in Venkataramanan and Lin (2013) and show that if the Lyapunov function is scale-invariant and sub-additive, a policy that performs steepest descent on this Lyapunov function is exponent optimal. In Section 5.4 we prove that SMW policies perform steepest descent on the policy-dependent Lyapunov functions and are hence exponent optimal. We also explicitly characterize the optimal exponent, the most likely sample paths leading to demand dropping, and the critical subsets.

5.1 Fluid Sample Paths and Fluid Limits

For any stationary dispatch policy $U \in \mathcal{U}$ defined in Section 2, we define the scaled demand and queue-length sample paths (extending (5) to the latter) by

$$\bar{\mathbf{A}}^K_{j'k}[t] \triangleq \frac{1}{K}\mathbf{A}^K_{j'k}[Kt]\,, \quad \bar{\mathbf{X}}^{K,U}_i[t] \triangleq \frac{1}{K}\mathbf{X}^{K,U}_i[Kt]\,, \tag{13}$$

for $t = 0, 1/K, 2/K, \ldots$. For other $t \geq 0$, we define $\bar{\mathbf{A}}^K_{j'k}[t]$ and $\bar{\mathbf{X}}^{K,U}_i[t]$ by linear interpolation. Note that for a fixed policy (with specified tie-breaking rules), each given demand sample path and initial state uniquely determines the state sample path. We denote this correspondence by $\Psi^{K,U} : (\bar{\mathbf{A}}^K[\cdot], \bar{\mathbf{X}}^{K,U}[0]) \mapsto \bar{\mathbf{X}}^{K,U}[\cdot]$.

To obtain a large deviation result, we need to look at the demand process and the queue-length process in the fluid scaling, as captured in (13). We take the standard approach of *fluid sample paths* (FSP) (see Stolyar 2003, Venkataramanan and Lin 2013).

Definition 3 (Fluid sample paths)**.** *We call a pair* $(\bar{\mathbf{A}}[\cdot], \bar{\mathbf{X}}^U[\cdot])$ *a fluid sample path (under stationary policy U) if there exists a sequence* $(\,\bar{\mathbf{A}}^K[\cdot], \bar{\mathbf{X}}^{K,U}[0], \Psi^{K,U}(\bar{\mathbf{A}}^K[\cdot], \bar{\mathbf{X}}^{K,U}[0])\,)$ *where* $\bar{\mathbf{A}}^K[\cdot]$ *are scaled demand sample paths and* $\bar{\mathbf{X}}^{K,U}[0] \in \Omega$*, such that it has a subsequence which converges to* $(\bar{\mathbf{A}}[\cdot], \bar{\mathbf{X}}^U[0], \bar{\mathbf{X}}^U[\cdot])$ *uniformly on* $[0, T]$*.*

In short, FSPs include both typical and atypical sample paths. Recall Fact 1, which gives the likelihood for an unlikely event to occur based on the most likely fluid sample path that causes the event. Accordingly, the large deviations analysis in Section 5.4 will identify the most likely FSP that leads to demand dropping.

Fluid limits are fluid sample paths that characterize *typical* system behavior, as they are the formal limits in the Functional Law of Large Numbers (Dai 1995).

Definition 4 (Fluid limits)**.** *We call a pair* $(\bar{\mathbf{A}}[\cdot], \bar{\mathbf{X}}^U[\cdot])$ *a fluid limit (under stationary policy U) if (i) the pair* $(\bar{\mathbf{A}}[\cdot], \bar{\mathbf{X}}^U[\cdot])$ *is a fluid sample path; (ii) we have* $\bar{\mathbf{A}}_{j'k}[t] = \phi_{j'k}t$*, for all* $j' \in V_D, k \in V_S$ *and all* $t \geq 0$*.*

5.2 A Family of Lyapunov Functions

Lyapunov functions are a useful tool for analyzing complex stochastic systems. In open queuing networks the ideal state is one in which all queues are empty, and correspondingly, the sum of squared queue lengths Lyapunov function is a popular choice (Tassiulas and Ephremides 1992, Eryilmaz and Srikant 2012, etc.); others have also used piecewise linear Lyapunov functions (Bertsimas et al. 2001, Venkataramanan and Lin 2013, etc.). Since our setting is a closed queueing network, we instead need to define a novel family of piecewise linear Lyapunov functions, parameterized by the desired state α, such that the function achieves its minimum at α.

Definition 5. *For each* $\alpha \in \mathrm{relint}(\Omega)$*, define Lyapunov function* $L_\alpha(\mathbf{x}) : \Omega \to [0, 1]$ *as* $L_\alpha(\mathbf{x}) \triangleq 1 - \min_i \frac{x_i}{\alpha_i}$*.*

The intuition behind our definition is as follows. The Lyapunov function value is jointly determined by the desired state α of the system (under some policy) and the objective, and can be interpreted as the energy of the system at each state. The desired state should have minimum energy, and the most undesirable states should have maximum energy. In our case the boundary $\partial\Omega$ of Ω is most undesirable since demand dropping only happens there, and correspondingly, $L_\alpha(\mathbf{x}) = 1$ for $\mathbf{x} \in \partial\Omega$, whereas $L_\alpha(\alpha) = 0$ as we want. In general, for $\mathbf{x} \in \Omega$, $L_\alpha(\mathbf{x})$ is one minus the smallest scaled queue length, given scaling factors α. See Figure 4 for an illustration.

These functions also have the following properties which are key in the analysis:

Lemma 1 (Key properties of $L_\alpha(\cdot)$). *For $L_\alpha(\cdot)$ with $\alpha \in \mathrm{relint}(\Omega)$, we have:*

1. *Scale-invariance (about α). $L_\alpha(\alpha + c\Delta\mathbf{x}) = cL_\alpha(\alpha + \Delta\mathbf{x})$ for any $c > 0$ and $\Delta\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{1}^T\Delta\mathbf{x} = 0$ and $\alpha + \Delta\mathbf{x} \in \Omega, \alpha + c\Delta\mathbf{x} \in \Omega$.*
2. *Sub-additivity (about α). $L_\alpha(\alpha+\Delta\mathbf{x}+\Delta\mathbf{x}') \le L_\alpha(\alpha+\Delta\mathbf{x})+L_\alpha(\alpha+\Delta\mathbf{x}')$ for any $\Delta\mathbf{x}, \Delta\mathbf{x}' \in \mathbb{R}^n$ such that $\mathbf{1}^T\Delta\mathbf{x} = \mathbf{1}^T\Delta\mathbf{x}' = 0$ and $\alpha + \Delta\mathbf{x} + \Delta\mathbf{x}', \alpha + \Delta\mathbf{x}, \alpha + \Delta\mathbf{x}' \in \Omega$.*

The proof of Lemma 1 is in Appendix B.

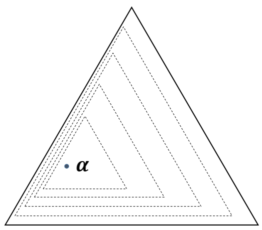

Figure 4: Sub-level sets of L_α when $|V_S| = |V_D| = 3$. State space Ω is the probability simplex in $\mathbb{R}^3$, and its boundary coincides with $\{\mathbf{x} : L_\alpha(\mathbf{x}) = 1, \mathbf{1}^T\mathbf{x} = 1\}$.

5.3 Sufficient Conditions for Exponent Optimality

In this section, we provide a converse bound on the exponent for any stationary policy $U \in \mathcal{U}$, and derive sufficient conditions for a policy to achieve this bound.

We use the intuition from differential games (see, e.g., Atar et al. 2003) to informally illustrate the interplay between the control and the most likely sample path leading to demand drop. Consider a zero-sum game between the adversary (nature) who chooses the fluid-scale demand arrival process $\bar{\mathbf{A}}[\cdot]$, and the controller who decides the assignment rule U, where the adversary minimizes the large-deviation "cost" of a demand sample path that leads to demand drop. Specifically, the adversary's cost for a demand sample path $\bar{\mathbf{A}}[\cdot]$ is the rate function defined in (8), i.e., the exponent. The converse bound we will obtain next will correspond to the adversary playing first and choosing the minimum cost *time-invariant* demand sample path that ensures demand drop. The following pleasant surprises will emerge subsequently: (i) we will find an equilibrium in pure strategies to the aforementioned zero-sum game, (ii) the converse will turn out to be tight, i.e., the adversary's equilibrium demand sample path will be time invariant, (iii) the controller's equilibrium assignment strategy will be an SMW policy with specific α (this simple policy will satisfy the sufficient conditions for achievability we will state immediately after our converse, in Proposition 3).

We provide a policy-independent upper bound of the exponent that only depends on the starting state. First, define

$$\mathcal{X}_{\mathbf{f}} \triangleq \left\{ \Delta\mathbf{x} \;\middle|\; \begin{array}{ll} \Delta x_i = \sum_{j'\in V_D} f_{j'i} - \sum_{j'\in\partial(i)} d_{ij'} \left(\sum_{k\in V_S} f_{j'k}\right), & \forall i \in V_S \\ \sum_{i\in\partial(j')} d_{ij'} = 1,\ d_{ij'} \geq 0, & \forall i \in V_S, j' \in V_D \end{array} \right\}, \tag{14}$$

which is the attainable change of (normalized) state in unit time, given that the average demand arrival rate during this period is $\mathbf{f}$ and assuming no demand is dropped. Then given starting state α, the attainable states at time T belong to $\alpha + T\mathcal{X}_{\mathbf{f}}$, if no demand is dropped during $[0, T]$ and the average demand arrival rate is $\mathbf{f}$. We obtain an upper bound on the demand-drop exponent by considering the most likely $\mathbf{f}$ and T such that $\alpha + T\mathcal{X}_{\mathbf{f}}$ is pushed out of state space Ω, no matter the assignment rule $\mathbf{d}$ used by the controller. Because the true state must lie in Ω, there must be demand drop during $[0, T]$.

Lemma 2 (Point-wise converse bounds of exponent)**.** *For any stationary policy $U \in \mathcal{U}$ and any $\epsilon > 0$, there exists $\alpha \in \mathrm{relint}(\Omega)$ such that (the subscript "CB" stands for "Converse Bound")*

$$-\liminf_{K\to\infty} \frac{1}{K} \log \mathbb{P}_{\mathrm{o}}^{K,U} \leq \gamma_{\mathrm{CB}}(\alpha) + \epsilon, \tag{15}$$

$$\text{where, for } \Lambda^*(\cdot) \text{ given by (6),} \quad \gamma_{\mathrm{CB}}(\alpha) \triangleq \inf_{\mathbf{f}: v_\alpha(\mathbf{f})>0} \frac{\Lambda^*(\mathbf{f})}{v_\alpha(\mathbf{f})}, \quad \text{and} \quad v_\alpha(\mathbf{f}) \triangleq \min_{\Delta\mathbf{x}\in\mathcal{X}_{\mathbf{f}}} L_\alpha(\alpha + \Delta\mathbf{x}).$$

In particular, $-\liminf_{K\to\infty}\frac{1}{K}\log\mathbb{P}_{\mathrm{o}}^{K,U}\le\sup_{\alpha\in\mathrm{relint}(\Omega)}\gamma_{\mathrm{CB}}(\alpha)$.

Here $v_\alpha(\mathbf{f})$ is the minimum rate of increase of $L_\alpha(\cdot)$ under demand $\mathbf{f}$, no matter the assignment rule $\mathbf{d}$. So, starting at α, the state hits Ω and demand is dropped in time at most $1/v_\alpha(\mathbf{f})$ under $\mathbf{f}$, implying a demand-drop exponent of at most $\frac{\Lambda^*(\mathbf{f})}{v_\alpha(\mathbf{f})}$. The upper bound $\gamma_{\mathrm{CB}}(\alpha)$ follows from minimizing over $\mathbf{f}$. Informally, the α in (15) is the most frequently visited (normalized) "resting" state in steady state under U. The proof of Lemma 2 is in Appendix C.

The following proposition provides sufficient conditions for a policy to achieve the converse bound exponent $\gamma_{\mathrm{CB}}(\alpha)$. The conditions are requirements on the time derivative of $L_\alpha(\bar{\mathbf{X}}^U[t])$.

Proposition 3 (Sufficient conditions). *Fix $\alpha\in\mathrm{relint}(\Omega)$. Let $U\in\mathcal{U}$ be a stationary, non-idling policy. Suppose that for each regular point t, the following hold:*

1. *(Steepest descent). For any demand fluid sample path $\bar{\mathbf{A}}[\cdot]$, we have*

$$\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t])=\inf_{U'\in\mathcal{U}_{\mathrm{ni}}}\left\{\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^{U'}[t])\,\middle|\,\bar{\mathbf{X}}^{U'}[t]=\bar{\mathbf{X}}^U[t]\right\},$$

   for corresponding queue-length sample paths, where $\mathcal{U}_{\mathrm{ni}}$ is the set of non-idling policies;

2. *(Negative drift). There exists $\eta>0$ and $\epsilon>0$ such that for all FSPs $(\bar{\mathbf{A}}[\cdot],\bar{\mathbf{X}}^U[\cdot])$ satisfying $\frac{d}{dt}\bar{\mathbf{A}}[t]\in B(\phi,\epsilon)$ and $\bar{\mathbf{X}}[t]\neq\alpha$, we have $\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t])\le-\eta$. Here $B(\phi,\epsilon)$ is a ball with radius ϵ centered on the typical demand arrival rate ϕ. In words, we require the policy to have negative Lyapunov drift for near typical demand arrival rate, as long as the current state is not α. This property forces the state to return to α.*

Then we have $-\limsup_{K\to\infty}\frac{1}{K}\log\mathbb{P}_{\mathrm{p}}^{K,U}=-\liminf_{K\to\infty}\frac{1}{K}\log\mathbb{P}_{\mathrm{o}}^{K,U}=\gamma_{\mathrm{CB}}(\alpha)$.

The full proof of Proposition 3 is quite technical and is included in Appendix D, but the key idea is straightforward. Given starting state α, the (i) steepest descent property of U and (ii) the scale-invariance and sub-additivity of $L_\alpha(\cdot)$, together ensure that the speed at which $L_\alpha(\cdot)$ increases under U cannot exceed the minimum speed $v_\alpha(\mathbf{f})$ in the converse construction (Lemma 2) for $\mathbf{f}\triangleq\frac{d}{dt}\bar{\mathbf{A}}[t]$. As a result, the demand drop exponent under U is no larger than $\gamma_{\mathrm{CB}}(\alpha)$. Mathematically,

$$\begin{aligned}&\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t])\\&=\inf_{U'\in\mathcal{U}}\left\{\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^{U'}[t])\,\middle|\,\frac{d}{dt}\bar{\mathbf{A}}'[t]=\mathbf{f}\right\}&&\text{(steepest descent)}\end{aligned}$$

$$
\begin{aligned}
&= \min_{\Delta\mathbf{x}\in\mathcal{X}_{\mathbf{f}}} \lim_{\Delta t\to 0} \frac{L_\alpha(\bar{\mathbf{X}}^U[t] + \Delta\mathbf{x}\Delta t) - L_\alpha(\bar{\mathbf{X}}^U[t])}{\Delta t} \\
&\le \min_{\Delta\mathbf{x}\in\mathcal{X}_{\mathbf{f}}} \lim_{\Delta t\to 0} \frac{L_\alpha(\alpha + \Delta\mathbf{x}\Delta t)}{\Delta t} && \text{(sub-additivity, Lemma 1)} \qquad (16) \\
&= \min_{\Delta\mathbf{x}\in\mathcal{X}_{\mathbf{f}}} L_\alpha(\alpha + \Delta\mathbf{x}) = v_\alpha(\mathbf{f})\,. && \text{(scale-invariance, Lemma 1)}
\end{aligned}
$$

Faced with a policy satisfying the above sufficient conditions, the adversary wants to force equality in (16) by forcing the queue-length sample path $\bar{\mathbf{X}}^U$ to go *radially* outward starting at α. This is why our converse in Lemma 2 based on a time invariant demand arrival process will turn out to be tight. We will formalize this intuition in Section 5.4 and explicitly characterize the most likely demand FSP forcing demand dropping.

5.4 Optimality of SMW Policies, Explicit Exponent, and Critical Subsets

In this section, we verify that SMW policies satisfy the sufficient conditions in Proposition 3. In doing so, we reveal the critical subset structure of the most-likely sample paths for demand drop and derive the explicit exponent for SMW(α). Proofs for this section are in Appendix E.

Lemma 3 shows that the Lyapunov drift only depends on the shortest scaled queue lengths, and that SMW(α) successfully minimizes its use of supplies from these queues.

Lemma 3 (SMW(α) causes steepest descent)**.** *Let* $(\bar{\mathbf{A}}, \bar{\mathbf{X}}^U)$ *be any FSP under any non-idling policy* U *on* $[0,T]$*, and consider any* $\alpha \in \mathrm{relint}(\Omega)$*. For a regular* $t \in [0,T]$*, define:*

$$
\begin{aligned}
S_1(\bar{\mathbf{X}}^U[t]) &\triangleq \left\{ k \in V_S : k \in \mathrm{argmin} \frac{\bar{\mathbf{X}}^U_k[t]}{\alpha_k} \right\}, \\
S_2\left(\bar{\mathbf{X}}^U[t], \frac{d}{dt}\bar{\mathbf{X}}^U[t]\right) &\triangleq \left\{ k \in S_1(\bar{\mathbf{X}}^U[t]) : k \in \mathrm{argmin} \frac{d}{dt}\frac{\bar{\mathbf{X}}^U_k[t]}{\alpha_k} \right\}.
\end{aligned}
$$

All the derivatives are well defined since t *is regular. We have*

$$
\frac{d}{dt} L_\alpha(\bar{\mathbf{X}}^U[t]) = -\frac{d}{dt}\frac{\bar{\mathbf{X}}^U_k[t]}{\alpha_k} \quad \text{for any } k \in S_2(\bar{\mathbf{X}}^U[t]) \tag{17}
$$

$$
\ge -\frac{1}{\mathbf{1}_{S_2}^{\mathrm{T}}\alpha} \left(\sum_{j'\in V_D, k\in S_2} \frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t] - \sum_{j'\in V_D : \partial(j')\subseteq S_2, k\in V_S} \frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t] \right). \tag{18}
$$

Inequality (18) holds with equality under SMW(α)*, i.e.,* SMW(α) *satisfies the steepest descent property in Proposition 3.*

In Lemma 4, we prove that SMW(α) satisfies the negative drift property. In particular, the drift η is related to the Hall's gap of the network; see Appendix E for details.

Lemma 4 (SMW(α) satisfies negative drift)**.** *For any $\alpha \in \mathrm{relint}(\Omega)$, under Assumptions 1 and 2, the policy* SMW*(α) satisfies the negative drift condition in Proposition 3.*

Combining Proposition 3 with Lemmas 1, 3 and 4, we immediately deduce that SMW(α) achieves the best possible exponent given resting state α.

Corollary 1. *For any $\alpha \in \mathrm{relint}(\Omega)$, we have $-\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}_{\mathrm{p}}^{K,\mathrm{SMW}(\alpha)} = \gamma_{\mathrm{CB}}(\alpha)$.*

We argued in Section 5.3 that the most likely queue-length sample path leading to demand drop with initial state α should be radial: when the controller chooses an exponent-optimal policy, the adversary picks a constant arrival rate $\mathbf{f}$ such that the sample path of queue lengths is radial starting at α, and the Lyapunov function increases at a constant rate. From Lemma 3 we see that the rate at which the Lyapunov function increases depends on the (scaled) inflow and outflow rate of supply in each subset. Since the most likely queue-length sample path is radial, this sample path should drain the supply of one subset (the critical subset), and that subset will determine the demand drop exponent. We first obtain the explicit expression of $\gamma_{\mathrm{CB}}(\alpha)$ and the most likely demand FSP forcing demand drop.

Lemma 5. *Recall the definitions of $\mathcal{J}$ in* (10) *and B_J, λ_J and μ_J in* (11)*. For any $\alpha \in \mathrm{relint}(\Omega)$, we have $\gamma_{\mathrm{CB}}(\alpha) = \gamma(\alpha) = \min_{J\in\mathcal{J}} B_J \log(\lambda_J/\mu_J)$. Moreover, the infimum in the definition of $\gamma_{\mathrm{CB}}(\alpha)$ in Lemma 2 is achieved by the following $\mathbf{f}^*$, for any $J^* \in \mathrm{argmin}_{J\in\mathcal{J}} B_J \log(\lambda_J/\mu_J)$,*

$$f^*_{j'k} \triangleq \begin{cases} \phi_{j'k}\lambda_{J^*}/\mu_{J^*} & \text{for } j' \in J^*, k \notin \partial(J^*)\,, \\ \phi_{j'k}\mu_{J^*}/\lambda_{J^*} & \text{for } j' \notin J^*, k \in \partial(J^*)\,, \\ \phi_{j'k} & \textit{otherwise}\,. \end{cases} \tag{19}$$

Remark 5 (Critical Subset Property)**.** *Lemma 5 provides the most likely demand sample path that leads to demand dropping under any dispatch policy that is exponent optimal, starting at state*[8] *α. We observe the critical subset property: (Adversary's strategy) For each starting state $\alpha \in \mathrm{relint}(\Omega)$, there is (are) corresponding critical subset(s) $J^* \in \mathrm{argmin}_{J\in\mathcal{J}} B_J \log(\lambda_J/\mu_J)$,*

[8]Remark 5 applies to demand dropped over a (long) finite horizon given starting state α. SMW(α) further forces the state to return to α (negative drift), so our observations carry over to the steady state as well under that policy.

such that the most likely demand sample path causing demand dropping drains a critical subset. (Controller's strategy) If the current state $\mathbf{x}$ *is on the most likely sample path forcing demand dropping in critical subset* J^* *starting at* α*, an exponent optimal policy (for given* α*) will maximally protect* J^* *at* $\mathbf{x}$*. Lemma 3 tells us that SMW*(α) *is such a policy.*

We can now prove the main theorem.

Proof of Theorem 1. Proof of achievability: for $\alpha \in \mathrm{relint}(\Omega)$, by Corollary 1 we know the pessimistic exponent of SMW(α) is equal to $\gamma_{\mathrm{CB}}(\alpha)$, which also implies that the optimistic exponent is equal to $\gamma_{\mathrm{CB}}(\alpha)$ by Lemma 2. By Lemma 5 we have $\gamma_{\mathrm{CB}}(\alpha) = \gamma(\alpha)$, which concludes the achievability proof.

The last sentence of Lemma 2 contains the converse result. $\square$

We conclude with a result that "vanilla" MaxWeight, i.e., SMW with identical scaling factors $\alpha = \frac{1}{n}\mathbf{1}$, achieves demand-drop exponent that is no smaller than $\frac{1}{n}$ of γ^*.

Proposition 4. *Suppose Assumption 2 holds, then we have* $\gamma\left(\frac{1}{n}\mathbf{1}\right) \geq \frac{1}{n}\gamma^*$.

6 Numerical Experiments

In this section, we use data from NYC Taxi & Limousine Commission and Google Maps to simulate SMW-based dispatch policies in an environment that resembles the real-world ride-hailing system in Manhattan, New York City. Our theoretical model made several simplifying assumptions (Section 2):

1. Service is *instantaneous* (i.e., vehicles travel to their destination with no delay).
2. Pickup is *instantaneous* (i.e., vehicles travel to matched customers with no delay).
3. The objective is system performance (fraction of dropped demand) *in steady state.*

We relax these assumptions one by one in our numerical experiments. We study three settings: (i) steady state performance with *Service times* (Section 6.2); (ii) steady state performance with *Service+Pickup times* (Section 6.3); and (iii) *Transient* performance with *Service+Pickup times* (Section 6.4). For the second and third settings, we modify SMW policies heuristically to incorporate pickup times. In each case, we assume a minimal realistic number of free cars

K by letting the number of cars in the system be only slightly (3% or less) above the "fluid requirement" (see Appendix F for a formal definition of the fluid requirement) to meet demand.

A highlight of SMW policies is that they are a simple family of policies with a manageable number of parameters (one per location). We describe how the scaling parameters α can be chosen in a practical setting using simulation-based optimization. Our theoretical results guarantee that this family of policies contains a member which is exponent optimal in the (huge) space of all possible policies, so we expect that there is some SMW policy with good performance (optimistically, near optimal performance). From there, we proceed based on the intuition that non-zero service times and pickup times do not damage the effectiveness of the SMW family policies at managing the spatial distribution of supply. Finally, we conjecture that there is an SMW policy that has exponent optimal transient performance (see Section 7). Accordingly, we deploy SMW policies in the Transient with Service + Pickup times setting as well.

Summary of findings. Consistently across all three settings, we find that the vanilla MaxWeight policy, which requires no knowledge of the demand arrival rates, vastly outperforms static (fluid-based) control, dropping very little demand even with small K (just ~ 10 free cars per location, whereas the static policy has a lot more free cars to work with since it drops so much more demand). Furthermore, in each of the settings, the SMW policy obtained using simulation-based optimization further significantly outperforms vanilla MaxWeight. Encouragingly, the simulation-based optimal scaling factors α that we find in the Service time setting are similar to the theory-based optimal α, indicating robustness of our structural results to travel time.

6.1 The Data, Simulation Environment and Benchmark

Throughout this section, we use the following set of model primitives.

- *Graph topology.* We consider a 30-location model of Manhattan below 110-th street excluding Central Park (see Figure 5), based on Buchholz (2015). We let pairs of regions which share a non-trivial boundary be compatible with each other.
- *Demand arrival process, Pickup/service times, and number of cars.* Throughout this section,

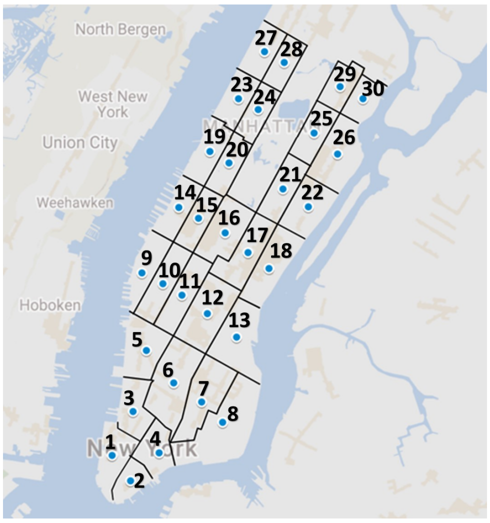

Figure 5: A 30 location model of Manhattan below 110-th street, excluding the Central Park. (Source: tessellation is based on Buchholz (2015), the figure is generated using Google Maps.)

we consider a stationary demand arrival rate that satisfies the CRP condition,[9] which is obtained by "symmetrizing" the decensored demand estimated in Buchholz (2015) (see Appendix F for a full description). We estimate travel times between location pairs using Google Maps, and use as a baseline the fluid requirement K_{fl} on number of cars needed to meet demand. We use K_{tot} (not K) to denote the total number of cars, and $K_{\mathrm{slack}} = K_{\mathrm{tot}} - K_{\mathrm{fl}}$ to denote the excess over the fluid requirement. Here K_{slack} is similar to the K in our theory since it is the average number of free cars assuming all demand is met.

Simulation Design. We consider the following simulation settings:

1. *Stationary performance with Service time.* We investigate steady state performance; steady state is reached in ∼1-2 hours under SMW policies.
2. *Stationary performance with Service+Pickup time.* Same as above.
3. *Transient performance with Service + Pickup time.* We investigate performance over a short horizon (below 2 hours) for different initial configurations.

Benchmark policy: fluid-based policy. The benchmark policy we consider is a static randomization based on the solution to the fluid problem (Banerjee et al. 2016, Ozkan and Ward 2016). See Appendix F for details.

[9]We leave the cases where demand is time-varying or violates CRP for future research. Our numerical study in Section 6.4 regarding transient performance may be seen as a first step towards the time-varying case, and for a setting where CRP is violated, we conjecture that a back-pressure type policy that generalizes SMW could work well.

Learning the optimal parameters. We use MATLAB's built-in `particleswarm` solver to learn the optimal SMW scaling parameters via simulation-based optimization in each setting.

6.2 Steady state with Service times

A preliminary simulation of the setting in our paper (i.e., pickup and service are both instantaneous) showed that under vanilla MaxWeight policy we only need $K_{\text{slack}} = 120$ to obtain a demand-drop rate below 1%, under SMW(α) with α defined in Theorem 1 the number further reduces to 80. However, the demand-drop rate stays above 5% under the fluid-based policy even when $K_{\text{slack}} = 200$.[10] We then proceeded to simulate the Service time setting, and obtained similarly encouraging results. In this setting, the average trip time is 13.2 minutes, and the fluid requirement is $K_{\text{fl}} = 7,061$ cars.

Results. The simulation results on performance[11] are shown in Figure 6, and the theoretical and learned α are shown in Figure 7. Figure 6 confirms that SMW policies including vanilla MaxWeight outperform the fluid-based policy; in fact only $K_{\text{slack}} = 100$ extra cars ($< 1.5\%$ of K_{tot}, or < 4 free cars per location on average if all demand is met) in the system lead to a negligible fraction of demand dropped. The demand dropping probability decays rapidly with K_{slack} under SMW policies, while it decays much slower under the fluid-based policy. SMW with parameters chosen based on Theorem 1 performs nearly as well as the learned SMW policy, despite small $K_{\text{slack}} = 100$. Figure 7 shows that the learned α is very similar to the theoretically optimal α structurally. Both policies allocate larger parameters (i.e., give more protection to the supply) in the Upper West Side area which has a small Hall's gap.

6.3 Steady state with Service and Pickup times

In the following experiment we further incorporate pickup times. The average pickup time is 5.5 minutes, and the fluid requirement increases to $K_{\text{fl}} = 10,002$ cars. Our objective here is to show that SMW policies can be heuristically adapted to more general settings, and retain their good performance. We propose the following SMW-based heuristic policy. Intuitively, pickup times need to be taken into consideration when making dispatch decisions, because every minute spent

[10] The results remain similar when service time is included, hence we only include the graph of the latter case.
[11] We also tested stochastic service times and found no significant difference in performance.

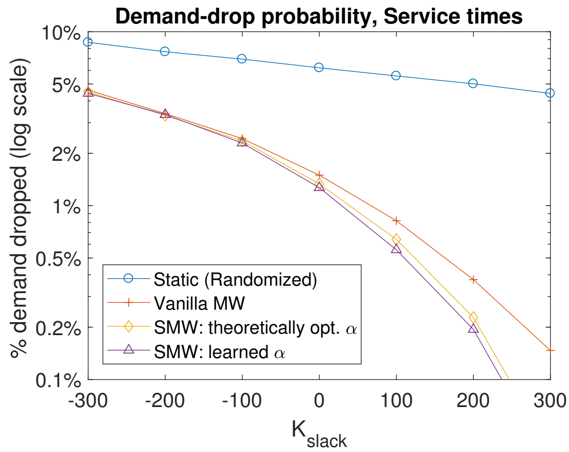

Figure 6: *Service times* setting: Stationary demand-drop probability under the static fluid-based policy, vanilla MaxWeight policy, SMW policy with theoretically optimal α, and SMW policy with learned α. Note that the y-axis is in log-scale. Here $K_{\text{fl}} = 7,061$. The plots indicate significant separation between fluid and SMW policies at all values of K, and separation between vanilla MaxWeight and optimized SMW. For each data point we run 200 trials and take the average.

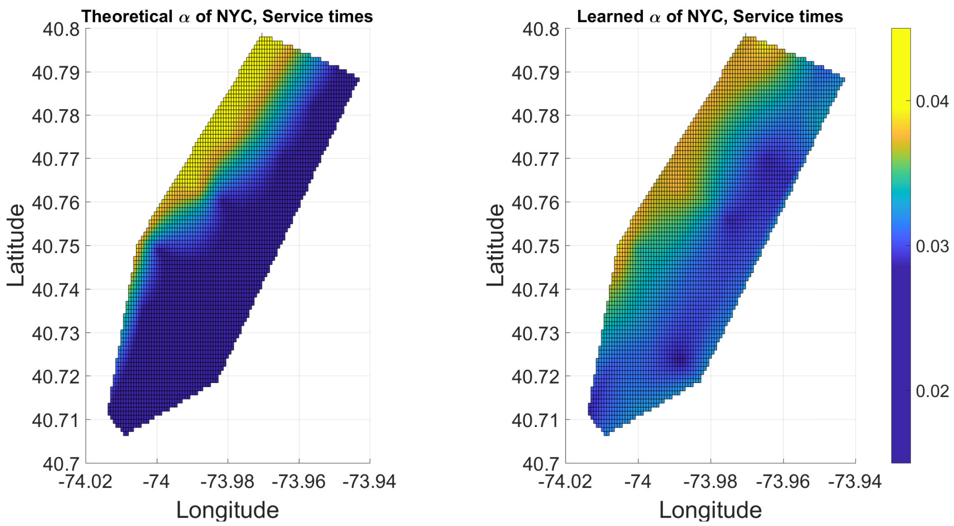

Figure 7: *Service times* setting: Theoretically optimal α derived from Theorem 1 (left) and the α learned via simulation-based optimization (right), both for the NYC dataset with $K_{\text{slack}} = 200$. Darker shades indicate smaller values of α_i, while lighter shades correspond to larger values.

on picking up a customer leads to an opportunity cost. We consider policies of the following form. When demand arrives at location j, dispatch from

$$\text{argmax}_{i \in \partial(j)} \frac{x_i}{\alpha_i} - \beta D_{ij},$$

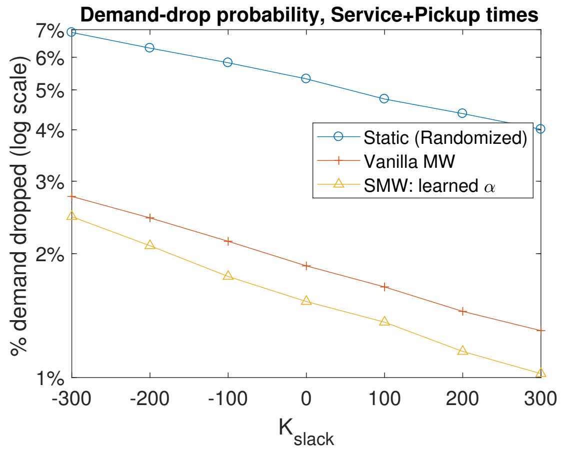

Figure 8: *Service+Pickup times* setting: Stationary demand-drop probability under the fluid-based policy, the vanilla MaxWeight policy, and the SMW policy with α learned via simulation optimization. Here $K_{\mathrm{fl}} = 10,002$ cars. For each data point we average over 200 trials.

where x_i is the number of free cars at i, and D_{ij} is the pickup time between i and j. In addition to scaling parameters α, we have an additional parameter β which captures the importance given to pickup delay in making dispatch decisions.

Results. Simulation results are shown in Figure 8. We observe that the SMW-based policies including vanilla MaxWeight significantly outperform the fluid-based policy. A few hundred extra cars ($< 3\%$ of K_{tot}) in the system suffice to ensure that only $\sim 1\%$ of demand is dropped.

6.4 Transient Behavior with Service and Pickup times

In the last experiment, we consider transient behavior instead of steady state performance. We fix K_{slack} to be 200. For initial configurations, we sample 4 initial queue-length vectors uniformly from the simplex $\{\mathbf{x} : x_1 + \cdots + x_{30} = 200\}$, and the cars initially in transit are based on picking up all demand that arose in the last hour. For each initial state we consider 4 time horizons: 0.5, 1, 1.5 and 2 hours. We learn the optimal SMW parameters for each initial state and time horizon pair to minimize the fraction of demand dropped and then compare the performance of SMW policies, vanilla MaxWeight and the fluid-based policy. The results are shown in Figure 9. It turns out that SMW policies outperform the fluid-based policy by an even larger margin in this case since they are able to quickly (in under an hour) spread the supply out across locations.

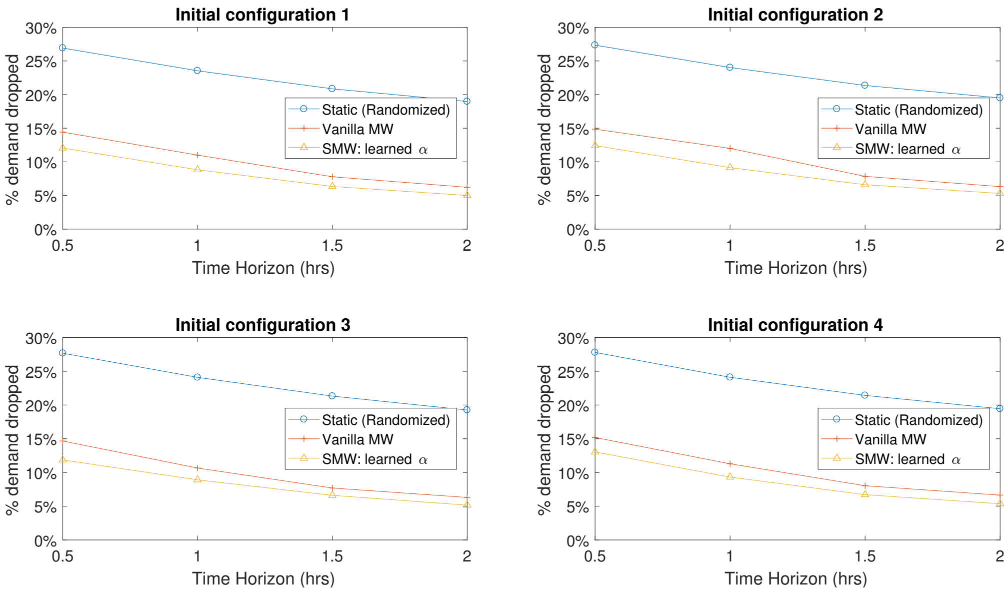

Figure 9: *Transient Performance with Service+Pickup times*: The plots show the demand-drop probability under the fluid-based policy, the vanilla MaxWeight policy, and the SMW policy with learned α, with 4 different initial configurations, chosen randomly on the simplex. We fix $K_{\text{slack}} = 200$, and consider time horizons ranging from 0.5 to 2 hours. For each data point we run 200 trials and take the average.

7 Conclusions and Future Directions

In this paper we study state-dependent assignment control of a shared transportation system modeled as a closed queueing network. We introduce a family of state-dependent assignment policies called Scaled MaxWeight (SMW) and prove that they have superior performance guarantees in terms of maximizing throughput, comparing with state-independent policies. In particular, we construct an SMW policy that (almost) achieves the optimal large deviation rate of decay of demand dropping. Our analysis also uncovers the structure of the problem: given system state, demand dropping is most likely to happen within a state-dependent critical subset of locations. The optimal SMW policy protects all subsets simultaneously. SMW policies are simple and explicit, and hence have the potential to influence practice.

Before closing, we point out several interesting directions for future theoretical research (note that the below concerns do not hinder practitioners who want to execute our suggestion of choosing scaling factors α for SMW using simulation-based optimization, see Section 6):

1. *Choosing α where ϕ is imperfectly known.* In practice, a key issue is knowledge of the true

arrival rates ϕ. We remark that if these rates are entirely unknown, but Assumption 2 holds, the platform can pick an SMW policy such as vanilla MaxWeight, and be sure to achieve exponential decay of the demand dropping probability (albeit with a suboptimal exponent). This is already an improvement over the state-independent control policies studied thus far Ozkan and Ward (2016), Banerjee et al. (2016) – in particular, any state-independent policy will drop an $\Omega(1)$ fraction of demand if there is any model misspecification whatsoever. If ϕ is not precisely known but is known to lie within some set, that setting may lend itself to a natural robust optimization problem of maximizing the demand dropping exponent, worse case over possible ϕ.

2. *Computational challenges.* A limitation of Theorem 1 is that it does not prescribe how to compute α. It simply specifies a concave maximization problem that must be solved, one where the objective is the minimum over an exponential number of linear functions. It seems plausible that structural properties of "realistic" G and ϕ may be exploited to make this problem tractable.

3. *General customer arrival process.* Because our analysis relies mostly on fluid sample paths and large deviations, the result may potentially be generalized to a setting where arrival processes are non-Poisson and spatially correlated. One would expect the arrival process information to enter our result through its moment generation function.

4. *Performance with service time.* As is demonstrated by simulation results (Section 6), SMW policies retain good performance with service times. Indeed, we can prove that in this case probability of demand drop still decreases exponentially as $K \to \infty$ under SMW policies, though it is unclear whether they are exponent-optimal.

5. *Relations to robust queueing.* The large deviation analysis in our paper can be interpreted as a zero-sum game between an adversary (nature) and the platform: the adversary tries to force demand dropping with minimum "cost" which is the large deviation rate function. Another interesting question would be: what is the worse case system performance if the adversary has a finite budget? The latter question is closely related to works on robust queueing (Bandi et al. 2015).

6. *Extension to bike-sharing systems.* In the context of bike sharing we may interpret the

assignment of supply units as suggestions to customers regarding where to pick up a bike. Further, the bike sharing setting may afford the platform the additional control lever of suggesting to customers where to drop off their bike, and has the additional wrinkle that both extremes are problematic — a station running out of empty docks, or of bikes. It would be interesting to extend our analysis incorporating those features.

7. *Relation to scrip systems.* Our model is related to scrip systems (e.g., see Johnson et al. 2014) which are also closed networks. Our findings might shed some light on the design of selections rules to match potential trade partners in scrip systems.
8. *Transient performance.* We have a conjecture on the SMW policies' transient performance: *For a given initial queue length vector and fixed time horizon, there exists an SMW policy that achieves the optimal demand-drop exponent among all state dependent policies.*

References

Adan, Ivo, Gideon Weiss. 2012. A loss system with skill-based servers under assign to longest idle server policy. *Probability in the Engineering and Informational Sciences* **26**(3) 307–321.

Adelman, Daniel. 2007. Price-directed control of a closed logistics queueing network. *Operations Research* **55**(6) 1022–1038.

Ata, Baris, Sunil Kumar. 2005. Heavy traffic analysis of open processing networks with complete resource pooling: asymptotic optimality of discrete review policies. *The Annals of Applied Probability* **15**(1A) 331–391.

Atar, Rami, Paul Dupuis, Adam Shwartz. 2003. An escape-time criterion for queueing networks: asymptotic risk-sensitive control via differential games. *Mathematics of Operations Research* **28**(4) 801–835.

Bandi, Chaithanya, Dimitris Bertsimas, Nataly Youssef. 2015. Robust queueing theory. *Operations Research* **63**(3) 676–700.

Banerjee, Siddhartha, Daniel Freund, Thodoris Lykouris. 2016. Pricing and optimization in shared vehicle systems: An approximation framework. *CoRR abs/1608.06819* .

Banerjee, Siddhartha, Yash Kanoria, Pengyu Qian. 2018. State dependent control of closed queueing networks. *Abstracts of the 2018 ACM International Conference on Measurement and Modeling of Computer Systems*. ACM, 2–4.

Bertsekas, Dimitri P. 1995. *Dynamic programming and optimal control*, vol. 1. Athena scientific Belmont, MA.

Bertsimas, Dimitris, David Gamarnik, John N Tsitsiklis. 2001. Performance of multiclass markovian queueing networks via piecewise linear lyapunov functions. *Annals of Applied Probability* 1384–1428.

Bimpikis, Kostas, Ozan Candogan, Daniela Saban. 2016. Spatial pricing in ride-sharing networks .

Blanchet, Jose. 2013. Optimal sampling of overflow paths in jackson networks. *Mathematics of Operations Research* **38**(4) 698–719.

Braverman, Anton, Jim G Dai, Xin Liu, Lei Ying. 2016. Empty-car routing in ridesharing systems. *arXiv preprint arXiv:1609.07219* .

Buchholz, Nicholas. 2015. Spatial equilibrium, search frictions and efficient regulation in the taxi industry. Tech. rep., Technical report, University of Texas at Austin.

Bumpensanti, Pornpawee, He Wang. 2018. A re-solving heuristic for dynamic resource allocation with uniformly bounded revenue loss. *arXiv preprint arXiv:1802.06192* .

Buỳić, Ana, Sean Meyn. 2015. Approximate optimality with bounded regret in dynamic matching models. *ACM SIGMETRICS Performance Evaluation Review* **43**(2) 75–77.

Cachon, Gerard P, Kaitlin M Daniels, Ruben Lobel. 2017. The role of surge pricing on a service platform with self-scheduling capacity. *Manufacturing & Service Operations Management* .

Caldentey, René, Edward H Kaplan, Gideon Weiss. 2009. Fcfs infinite bipartite matching of servers and customers. *Advances in Applied Probability* **41**(3) 695–730.

Chung, Kai Lai. 2001. *A course in probability theory*. Academic press.

Dai, Jim G. 1995. On positive harris recurrence of multiclass queueing networks: a unified approach via fluid limit models. *The Annals of Applied Probability* 49–77.

Dai, Jim G, Wuqin Lin. 2005. Maximum pressure policies in stochastic processing networks. *Operations Research* **53**(2) 197–218.

Dai, Jim G, Wuqin Lin. 2008. Asymptotic optimality of maximum pressure policies in stochastic processing networks. *The Annals of Applied Probability* **18**(6) 2239–2299.

Dembo, Amir, Ofer Zeitouni. 1998. *Large deviations techniques and applications*, *Application of Mathematics*, vol. 38. 2nd ed. Springer.

Désir, Antoine, Vineet Goyal, Yehua Wei, Jiawei Zhang. 2016. Sparse process flexibility designs: is the long chain really optimal? *Operations Research* **64**(2) 416–431.

Eryilmaz, Atilla, R Srikant. 2012. Asymptotically tight steady-state queue length bounds implied by drift conditions. *Queueing Systems* **72**(3-4) 311–359.

Gallego, Guillermo, Garrett Van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management science* **40**(8) 999–1020.

Gurvich, Itay, Ward Whitt. 2009. Scheduling flexible servers with convex delay costs in many-server service systems. *Manufacturing & Service Operations Management* **11**(2) 237–253.

Hall, Jonathan, Cory Kendrick, Chris Nosko. 2015. The effects of uber's surge pricing: A case study. *The University of Chicago Booth School of Business* .

Harrison, J Michael, Marcel J López. 1999. Heavy traffic resource pooling in parallel-server systems. *Queueing systems* **33**(4) 339–368.

Jasin, Stefanus, Sunil Kumar. 2012. A re-solving heuristic with bounded revenue loss for network revenue management with customer choice. *Mathematics of Operations Research* **37**(2) 313–345.

Johnson, Kris, David Simchi-Levi, Peng Sun. 2014. Analyzing scrip systems. *Operations Research* **62**(3) 524–534.

Jordan, William C, Stephen C Graves. 1995. Principles on the benefits of manufacturing process flexibility. *Management Science* **41**(4) 577–594.

Maguluri, Siva Theja, R Srikant. 2016. Heavy traffic queue length behavior in a switch under the maxweight algorithm. *Stochastic Systems* **6**(1) 211–250.

Mairesse, Jean, Pascal Moyal. 2016. Stability of the stochastic matching model. *Journal of Applied Probability* **53**(4) 1064–1077.

Majewski, Kurt, Kavita Ramanan. 2008. How large queue lengths build up in a jackson network .

Meyn, Sean P, RL Tweedie. 1994. State-dependent criteria for convergence of markov chains. *The Annals of Applied Probability* 149–168.

Ozkan, Erhun, Amy R Ward. 2016. Dynamic matching for real-time ridesharing .

Ross, Sheldon M. 1996. Stochastic processes. 1996.

Rudin, Walter. 1964. *Principles of mathematical analysis*, vol. 3. McGraw-hill New York.

Shi, Cong, Yehua Wei, Yuan Zhong. 2015. Process flexibility for multi-period production systems .

Stolyar, Alexander L. 2003. Control of end-to-end delay tails in a multiclass network: Lwdf discipline optimality. *Annals of Applied Probability* 1151–1206.

Stolyar, Alexander L. 2004. Maxweight scheduling in a generalized switch: State space collapse and workload minimization in heavy traffic. *The Annals of Applied Probability* **14**(1) 1–53.

Stolyar, Alexander L, Kavita Ramanan. 2001. Largest weighted delay first scheduling: Large deviations and optimality. *Annals of Applied Probability* 1–48.

Subramanian, Vijay G. 2010. Large deviations of max-weight scheduling policies on convex rate regions. *Mathematics of Operations Research* **35**(4) 881–910.

Talluri, Kalyan T, Garrett J Van Ryzin. 2006. *The theory and practice of revenue management*, vol. 68. Springer Science & Business Media.

Tassiulas, Leandros, Anthony Ephremides. 1992. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE transactions on automatic control* **37**(12) 1936–1948.

Venkataramanan, VJ, Xiaojun Lin. 2013. On the queue-overflow probability of wireless systems: A new approach combining large deviations with lyapunov functions. *IEEE Transactions on Information Theory* **59**(10) 6367–6392.

Waserhole, Ariel, Vincent Jost. 2016. Pricing in vehicle sharing systems: Optimization in queuing networks with product forms. *EURO Journal on Transportation and Logistics* **5**(3) 293–320.

Appendix

This technical appendix is organized as follows.

- Appendix A shows that the CRP condition and state-dependent control are necessary to achieve exponential decay of demand drop probability, including the proofs of Proposition 1 and Proposition 2.
- Appendix B discusses fluid sample paths in detail and establishes key properties of our Lyapunov functions, including the proof of Lemma 1.
- Appendix C includes the proof of Lemma 2, a converse bound on the demand drop exponent.
- Appendix D includes the proof of Proposition 3, containing sufficient conditions for a policy to achieve the optimal exponent.
- Appendix E shows that SMW satisfies the sufficient conditions above, and derives explicitly the optimal exponent and most-likely sample paths, including the proofs of Lemma 3, Lemma 4, Lemma 5 and Proposition 4.
- Appendix F explains the simulation settings in more detail.

A Necessity of CRP Condition and State-Dependent Control for Exponential Decay

A.1 Necessity of CRP Condition: Proof of Proposition 1

Proof of Proposition 1. There are two cases:

1. *Some inequalities in (9) are strictly reversed, i.e., there exists $J \subsetneq V_D$ s.t. $\sum_{i\in\partial(J)} \mathbf{1}^{\mathrm{T}}\phi_{(i)} < \sum_{j'\in J} \mathbf{1}^{\mathrm{T}}\phi_{j'}$.* Consider the following balance equation:

$$\begin{aligned}
&\#\{\text{demands arrive to } J \text{ in } [1, Kt] \text{ that are dropped}\} \\
=\ &\#\{\text{demands arrive to } J \text{ in } [1, Kt]\} - \#\{\text{demands arrive to } J \text{ in } [1, Kt] \text{ that are fulfilled}\} \\
\geq\ &\#\{\text{demands arrive to } J \text{ in } [1, Kt]\} - \#\{\text{supplies assigned from } \partial(J) \text{ in } [1, Kt]\} \\
\geq\ &\sum_{s=1}^{Kt} \mathbb{I}\{o[s] \in J\} - \sum_{s=1}^{Kt} \mathbb{I}\{d[s] \in \partial(J)\} - \#\{\text{initial supply in } \partial(J)\}\,.
\end{aligned}$$

   Divide both sides by K and let $K \to \infty$, by the strong law of large numbers, we have:

$$\liminf_{t\to\infty}\{\text{proportion of dropped demand in } [1, Kt]\} \geq \sum_{j'\in J} \mathbf{1}^{\mathrm{T}}\phi_{j'} - \sum_{i\in\partial(J)} \mathbf{1}^{\mathrm{T}}\phi_{(i)} > 0\,,$$

   hence a positive portion of demand will be dropped, and the exponent is 0.

2. *No inequality in (9) is strictly reversed but there exists $J \in \mathcal{J}$ such that $\sum_{i\in\partial(J)} \mathbf{1}^{\mathrm{T}}\phi_{(i)} = \sum_{j'\in J} \mathbf{1}^{\mathrm{T}}\phi_{j'}$.* Divide the discrete time periods into epochs with length MK^2, where

$$M \triangleq \frac{6}{\sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}}\,.$$

   Without loss of generality, consider the first epoch $[1, MK^2]$. Define random walk S_t with the following dynamics:

   - $S_0 = \mathbf{1}^{\mathrm{T}}_{\partial(J)}\mathbf{X}[0]$.
   - $S_{t+1} = S_t + 1$ if $o[s] \notin J, d[s] \in \partial(J)$.
   - $S_{t+1} = S_t - 1$ if $o[s] \in J, d[s] \notin \partial(J)$.
   - $S_{t+1} = S_t$ otherwise.

   It is not hard to see that if no demand is dropped during $[1, T]$ under some policy U, then S_t is an upper bound on the number of supply units in $\partial(J)$, namely, $\mathbf{1}^{T}_{\partial(J)}\mathbf{X}[t]$, for any $t \in [1, T]$. With this observation, we have:

$$\begin{aligned}
&\mathbb{P}\left(\text{some demand is dropped in } [1, MK^2]\right) \\
\geq\ &\mathbb{P}\left(S_{t'} = 0 \text{ for some } t' \in [1, MK^2]\right) \cdot \sum_{j'\in J, k\in V_S} \phi_{j'k}\,.
\end{aligned}$$

   The above is true because when the event on RHS happens, either (1) some demand is dropped before t', or (2) no demand is dropped before t', then since $0 = S_{t'} \geq \mathbf{1}^{T}_{\partial(J)}\mathbf{X}[t']$, any demand with origin in J is dropped at t'. For J of interest, note that S_t is an unbiased random walk, hence a martingale. It is easy to verify that

$$G_t = S_t^2 - 2t \sum_{j' \in J, k \notin \partial(J)} \phi_{j'k}$$

is also a martingale. Using the optional stopping theorem, we have the following bound for the first time τ when S_t hits $\left\{\mathbf{1}_{\partial(J)}^{\mathrm{T}}\mathbf{X}[0] - 2K, \mathbf{1}_{\partial(J)}^{\mathrm{T}}\mathbf{X}[0] + 2K\right\}$:

$$\mathbb{E}[\tau] = \frac{2K^2}{\sum_{j' \in J, k \notin \partial(J)} \phi_{j'k}}.$$

Note that S_t hitting $\left\{\mathbf{1}_{\partial(J)}^{\mathrm{T}}\mathbf{X}[0] - 2K\right\}$ implies that it hit 0 before, so we have

$$\begin{aligned}
&\mathbb{P}\left(S_{t'} = 0 \text{ for some } t' \in [1, MK^2]\right) \\
&= 1 - \mathbb{P}(\tau > MK^2) - \mathbb{P}(\tau \leq MK^2)\mathbb{P}\left(\mathbf{X}[\tau] = \mathbf{1}_{\partial(J)}^{\mathrm{T}}\mathbf{X}[0] + 2K \,\middle|\, \tau \leq MK^2\right) \\
&\geq 1 - \mathbb{P}(\tau > 3\mathbb{E}[\tau]) - \frac{1}{2} \\
&\geq \frac{1}{6},
\end{aligned}$$

where the last inequality follows from Markov's inequality. As a result, the probability that demand drop happens during any epoch of length MK^2 is lower bounded by a uniform constant under any U. Therefore we have

$$\mathbb{P}_{\mathrm{p}}^{K,U} \geq \mathbb{P}_{\mathrm{o}}^{K,U} = \Omega\left(\frac{1}{K^2}\right),$$

hence $\gamma_{\mathrm{p}}(U) = \gamma_{\mathrm{o}}(U) = 0$ for any U.

□

A.2 Necessity of State-dependent Control: Proof of Proposition 2

Proof of Proposition 2. Denote the probability mass function of distribution $u_{j'k}[t]$ by $u_{j'k}[t](\cdot)$. We first define an "augmented" policy $\tilde{\pi}$ for any state-independent policy π. Policy $\tilde{\pi}$ is also state independent with distribution $\tilde{u}_{j'k}[t]$, where:

$$\begin{aligned}
&\tilde{u}_{j'k}[t](i) = u_{j'k}[t](i) + \frac{1}{|\partial(j')|} u_{j'k}[t](\emptyset) \quad \text{for } i \in \partial(j'), \\
&\tilde{u}_{j'k}[t](\emptyset) = 0.
\end{aligned}$$

In the following analysis, we couple π and $\tilde{\pi}$ in such a way that if π dispatches from i to serve the t-th demand, then $\tilde{\pi}$ will do the same.

We first divide the discrete periods into epochs with length K^2. We will lower bound the probability of demand drop in any epoch. Without loss of generality, consider epoch $[1, K^2]$. Suppose $\mathbf{X}^{K,\pi}[0] = \mathbf{X}_0$. By Assumption 1, $\exists j' \in V_D$, $k \notin \partial(j') \subset V_S$ such that $\phi_{j'k} > 0$. Consider the following process S_t, which is the "virtual" net change of supply in $\partial(j')$:

- $S_0 = 0$.
- $S_{t+1} = S_t + 1$ if $d[t] \in \partial(j')$ and policy $\tilde{\pi}$ dispatches a vehicle from outside of $\partial(j')$ to serve it (regardless of whether there is available supply to dispatch).

- $S_{t+1} = S_t - 1$ if $d[t] \notin \partial(j')$ and policy $\tilde{\pi}$ dispatches a vehicle from $\partial(j')$ to serve it (regardless of whether there is available supply to dispatch).
- $S_{t+1} = S_t$ if otherwise.

We argue that

$$\mathbb{P}\left(\text{some demand is dropped in epoch } [1, K^2]\right) \geq \mathbb{P}\left(S_{K^2} + \mathbf{1}^T_{\partial(j')}\mathbf{X}_0 > K \text{ or } < 0\right), \tag{20}$$

$$\geq \mathbb{P}\left(|S_{K^2}| > K\right). \tag{21}$$

(20) holds because when the event on the left-hand-side (LHS) does not happen, the event on the right-hand-side (RHS) does not happen either. Suppose there is no demand drop during $[1, K^2]$ under policy π, then there is no demand drop under $\tilde{\pi}$ either (because of the coupling). Hence $S_{K^2} + \mathbf{1}^T_{\partial(j')}\mathbf{X}_0$ is exactly the number of supplies in $\partial(j')$ at K^2, which cannot exceed K or be negative. (21) holds because $\{|S_{K^2}| > K\} \Rightarrow \{S_{K^2} + \mathbf{1}^T_{\partial(j')}\mathbf{X}_0 > K \text{ or } < 0\}$.

Note that S_{K^2} is the sum of K^2 independent random variables Z_t, where $Z_t = S_t - S_{t-1}$. Here independence holds because we ignore demand drops in the definition of the process. Here Z_t has support $\{-1, 0, 1\}$ and satisfies:

$$\mathbb{P}(Z_t = -1) \geq \delta \triangleq \phi_{j'k} > 0\,, \tag{22}$$

where $k \notin \partial(j)$. There are two cases:

1. If $\mathbb{E}[S_{K^2}] \leq -\frac{K^2}{2}$, then for $K \geq 8$, we have

$$\begin{aligned}
&1 - \mathbb{P}\left(S_{K^2} \in [-K, K]\right) \\
&\geq 1 - \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \geq -K + \frac{K^2}{2}\right) \\
&\geq 1 - 2\exp\left(-\frac{K^2}{32}\right) \qquad \text{(Hoeffding's inequality, } -K + K^2/2 \geq K^2/4) \\
&\geq \frac{1}{2}\,.
\end{aligned}$$

   Plugging into (21) establishes that demand is dropped with likelihood at least 1/2.
2. If $\mathbb{E}[S_{K^2}] > -\frac{K^2}{2}$, then using linearity of expectation and simple algebra we obtain that the number of t's such that $\mathbb{E}[Z_t] \geq -\frac{3}{4}$ is at least $\frac{K^2}{7}$.
   Denote the set of these t's as $\mathcal{T}$. Hence

$$K^2 \geq \mathrm{Var}(S_{K^2}) = \sum_{t=1}^{K^2} \mathrm{Var}(Z_t) \geq \sum_{t \in \mathcal{T}} \mathrm{Var}(Z_t) \geq \frac{K^2}{7} \cdot \delta \left(1 - \frac{3}{4}\right)^2 = \frac{\delta}{102} K^2\,, \tag{23}$$

   using (22).
   Note from (21) that to show a constant lower bound of demand-drop probability on $[1, K^2]$, it suffices to derive a uniform upper bound on $\mathbb{P}\left(S_{K^2} \in [-K, K]\right)$ that is strictly smaller than 1. To this end, apply Theorem 7.4.1 in Chung (2001) (Berry-Esseen Theorem) to obtain:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \leq x\sqrt{\mathrm{Var}[S_{K^2}]}\right) - \Phi(x) \right| \leq \frac{\sum_{t=1}^{K^2} \mathbb{E}|Z_t - \mathbb{E}Z_t|^3}{\left(\mathrm{Var}[S_{K^2}]\right)^{3/2}} \leq 5000\delta^{-3/2}K^{-1}\,, \tag{24}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Denote $B(x,a) \triangleq [x-a, x+a]$. Note that there are two subcases (indexed 2(i) and 2(ii)):

$$[-K,K] \subset B\left(\mathbb{E}[S_{K^2}], 4K\right), \qquad [-K,K] \cap B\left(\mathbb{E}[S_{K^2}], 2K\right) = \emptyset\,.$$

In subcase 2(i),

$$\mathbb{P}\left(S_{K^2} \in [-K,K]\right) \le \mathbb{P}\left(S_{K^2} \in B\left(\mathbb{E}[S_{K^2}], 4K\right)\right)\,,$$

whereas in subcase 2(ii),

$$\mathbb{P}\left(S_{K^2} \in [-K,K]\right) \le 1 - \mathbb{P}\left(S_{K^2} \in B\left(\mathbb{E}[S_{K^2}], 2K\right)\right)\,.$$

Hence

$$\mathbb{P}\left(S_{K^2} \in [-K,K]\right) \le \max\left\{\mathbb{P}\left(S_{K^2} \in B\left(\mathbb{E}[S_{K^2}], 4K\right)\right), 1 - \mathbb{P}\left(S_{K^2} \in B\left(\mathbb{E}[S_{K^2}], 2K\right)\right)\right\}\,. \tag{25}$$

Use (24) and $\mathrm{Var}(S_{K^2}) \le K^2$ to obtain

$$\begin{aligned}
\mathbb{P}\left(S_{K^2} \in B\left(\mathbb{E}[S_{K^2}], 4K\right)\right) &\le \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \le \sqrt{\mathrm{Var}[S_{K^2}]}\frac{4K}{\sqrt{\mathrm{Var}[S_{K^2}]}}\right) \\
&\le 5000\delta^{-3/2}K^{-1} + \Phi\left(\frac{4K}{\sqrt{\mathrm{Var}[S_{K^2}]}}\right) \\
&\le 5000\delta^{-3/2}K^{-1} + \Phi\left(50\delta^{-1/2}\right)\,,
\end{aligned}$$

$$\begin{aligned}
1 - \mathbb{P}\left(S_{K^2} \in B\left(\mathbb{E}[S_{K^2}], 2K\right)\right) &= \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \le \sqrt{\mathrm{Var}[S_{K^2}]}\frac{-2K}{\sqrt{\mathrm{Var}[S_{K^2}]}}\right) \\
&\quad + \mathbb{P}\left(S_{K^2} - \mathbb{E}[S_{K^2}] \ge \sqrt{\mathrm{Var}[S_{K^2}]}\frac{2K}{\sqrt{\mathrm{Var}[S_{K^2}]}}\right) \\
&\le 10000\delta^{-3/2}K^{-1} + 2\Phi\left(\frac{-2K}{\sqrt{\mathrm{Var}[S_{K^2}]}}\right) \\
&\le 10000\delta^{-3/2}K^{-1} + 2\Phi\left(-2\right)\,.
\end{aligned}$$

Hence for $K > \max\left\{\frac{10000\delta^{-3/2}}{\bar{\Phi}\left(50\delta^{-1/2}\right)}, \frac{10000\delta^{-3/2}}{\frac{1}{2}-\Phi(-2)}\right\}$, plugging into (25) and then into (21), we obtain

$$\mathbb{P}(\text{some demand is dropped in } [1,K^2]) \ge \min\left\{\frac{1}{2}\bar{\Phi}\left(50\delta^{-1/2}\right), \frac{1}{2} - \Phi(-2)\right\} > 0\,.$$

Since we obtained a uniform lower bound on the likelihood of dropping demand in both cases, we conclude that the steady state demand-drop probability is $\Omega(1/K^2)$ as $K \to \infty$. $\square$

B Fluid Sample Paths and Lyapunov Functions

B.1 Existence of FSPs

The existence of FSPs under Definition 3 is guaranteed by the Arzelà-Ascoli theorem. Since all elements in the sequence $(\bar{\mathbf{A}}^K[\cdot], \bar{\mathbf{X}}^{K,U}[0], \Psi^{K,U}(\bar{\mathbf{A}}^K[\cdot], \bar{\mathbf{X}}^{K,U}[0]))$ are Lipschitz continuous with Lipschitz constant 1, it must have a subsequence that converges uniformly to a limit by Arzelà-Ascoli theorem. Moreover, uniform convergence passes the Lipschitz continuity to the limit (see Rudin 1964), hence all FSPs are Lipschitz continuous.

Also, one must take extra care when stating results involving FSP. Although for fixed $U \in \mathcal{U}$, each $\bar{\mathbf{A}}^K[\cdot]$ uniquely defines a queue-length sample path $\bar{\mathbf{X}}^{K,U}[\cdot]$, it is not necessarily true that the $\bar{\mathbf{A}}[\cdot]$ component in FSP uniquely defines the $\bar{\mathbf{X}}^U[\cdot]$ component. In other words, it is possible that for different $\bar{\mathbf{X}}_1^U[\cdot]$ and $\bar{\mathbf{X}}_2^U[\cdot]$, $(\bar{\mathbf{A}}[\cdot], \bar{\mathbf{X}}_1^U[\cdot])$ and $(\bar{\mathbf{A}}[\cdot], \bar{\mathbf{X}}_2^U[\cdot])$ are both FSPs. Contraction principle (see Dembo and Zeitouni 1998) can rule out such behaviors, but it is technically challenging to prove it for MaxWeight policies (see Subramanian 2010 for its proof under a different setting). As a result, we circumvent this technicality by considering *all* FSPs in the following proofs.

B.2 Properties of the Lyapunov Functions $L_\alpha(\mathbf{x})$

B.2.1 Scale-invariance and sub-additivity (about α): proof of Lemma 1

Proof of Lemma 1. (i) For $c > 0$, $\alpha \in \mathrm{relint}(\Omega)$, we have

$$L_\alpha(\alpha + c\Delta\mathbf{x}) = 1 - \min_i \frac{\alpha_i + c\Delta x_i}{\alpha_i} = -\min_i \frac{c\Delta x_i}{\alpha_i} = -c\min_i \frac{\Delta x_i}{\alpha_i} = cL_\alpha(\alpha + \Delta\mathbf{x})\,.$$

(ii) For $\alpha \in \mathrm{relint}(\Omega)$, we have

$$\begin{aligned} L_\alpha(\alpha + \Delta\mathbf{x} + \Delta\mathbf{x}') &= 1 - \min_i \frac{\alpha_i + \Delta x_i + \Delta x'_i}{\alpha_i} = -\min_i \frac{\Delta x_i + \Delta x'_i}{\alpha_i} \\ \leq\ -\min_i \frac{\Delta x_i}{\alpha_i} - \min_i \frac{\Delta x'_i}{\alpha_i} &= L_\alpha(\alpha + \Delta\mathbf{x}) + L_\alpha(\alpha + \Delta\mathbf{x}')\,. \end{aligned}$$

$\square$

B.2.2 Regularity properties

The following lemma is a collection of regularity properties of $L_\alpha(\mathbf{x})$ that are useful in the following proofs. The norm $||\cdot||$ is Euclidean norm unless stated otherwise.

Lemma 6. *For $\alpha \in \mathrm{relint}(\Omega)$ and $L_\alpha(\mathbf{x})$ specified in Definition 5, we have*

1. *$L_\alpha(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Omega$, and $L_\alpha(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \alpha$.*
2. *$L_\alpha(\mathbf{x})$ is globally Lipschitz on Ω, i.e.*

$$|L_\alpha(\mathbf{x}_1) - L_\alpha(\mathbf{x}_2)| \leq \frac{1}{\min_{i\in V_S}\alpha_i}||\mathbf{x}_1 - \mathbf{x}_2||_\infty\,.$$

3. *For all fluid sample paths $(\bar{\mathbf{A}}[\cdot], \bar{\mathbf{X}}[\cdot])$ (here $n = |V_S|$),*

$$\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}[t]) \leq \frac{1}{\min_{i\in V_S}\alpha_i}\,.$$

Proof of Lemma 6. Properties 1 is easy to verify hence we omit the proof.

For property 2, define $i_\alpha(\mathbf{x})$ as the smallest index in index set: $\operatorname{argmin}_{i \in V_S} \frac{x_i}{\alpha_i}$. It's not hard to see that $\mathbf{g}_\alpha(\mathbf{x}) \triangleq -\frac{1}{\alpha_{i_\alpha(\mathbf{x})}} \mathbf{e}_{i_\alpha(\mathbf{x})}$ is a sub-gradient of $L_\alpha(\cdot)$ at $\mathbf{x}$, where $\mathbf{e}_i$ is the i-th unit vector. Using the convexity of $L_\alpha(\cdot)$ (implied by the concavity of minimum function) we have

$$
\begin{aligned}
L_\alpha(\mathbf{x}_1) - L_\alpha(\mathbf{x}_2) &\geq \mathbf{g}_\alpha(\mathbf{x}_2)^T(\mathbf{x}_1 - \mathbf{x}_2)\,, \\
L_\alpha(\mathbf{x}_2) - L_\alpha(\mathbf{x}_1) &\geq \mathbf{g}_\alpha(\mathbf{x}_1)^T(\mathbf{x}_2 - \mathbf{x}_1)\,.
\end{aligned}
$$

Therefore using Hölder's inequality we have

$$
\begin{aligned}
|L_\alpha(\mathbf{x}_1) - L_\alpha(\mathbf{x}_2)| &\leq \Big(\max\left\{||\mathbf{g}_\alpha(\mathbf{x}_1)||_1, ||\mathbf{g}_\alpha(\mathbf{x}_2)||_1\right\}\Big) \cdot ||\mathbf{x}_1 - \mathbf{x}_2||_\infty \\
&\leq \frac{1}{\min_{i \in V_S} \alpha_i} \cdot ||\mathbf{x}_1 - \mathbf{x}_2||_\infty\,.
\end{aligned}
$$

For property 3, note that $\bar{\mathbf{X}}[t]$ is Lipschitz with Lipschitz constant 1 in the l^∞ norm since one demand arrives per period. Combined with 2, we conclude the proof. $\square$

C Point-wise Converse Bound: Proof of Lemma 2

Proof of Lemma 2. Step 1: Find the proper α. Fix a sequence of stationary policies. For each K, the system under policy U is a finite state Markov chain. The cardinality of this chain's state space is smaller than K^n. Since we are considering the optimistic exponent, let the K-th system start with a state that minimizes steady state demand drop among all initial states. Denote the stationary distribution of (normalized) states as $\pi^K(\cdot)$. Then there must exist a (normalized) state $\tilde{\mathbf{x}}^K$ such that $\pi(\tilde{\mathbf{x}}^K) \geq K^{-n}$. Take a subsequence $\{K_r\}$ of $\{K\}$ such that $\lim_{r\to\infty} \frac{1}{K_r} \log \mathbb{P}_{\mathrm{o}}^{K_r,U} = \liminf_{K\to\infty} \frac{1}{K} \log \mathbb{P}_{\mathrm{o}}^{K,U}$. By compactness of Ω, there must exist a further subsequence of $\{K_r\}$, denoted by $\{K_{r'}\}$, and $\alpha \in \Omega$ such that $\lim_{r'\to\infty} \tilde{\mathbf{x}}^{K_{r'}} = \alpha$.

For any $0 < \epsilon_1 < \frac{1}{2}\Big(\min_{j:\alpha_j>0} \alpha_j\Big)$, define $\tilde{\alpha} \in \operatorname{relint}(\Omega)$ such that

$$
\begin{aligned}
0 < \tilde{\alpha}_j &< \epsilon_1/2 \quad \text{for } j \text{ such that } \alpha_j = 0\,, \\
|\tilde{\alpha}_j - \alpha_j| &< \epsilon_1/2 \quad \text{for } j \text{ such that } \alpha_j > 0\,.
\end{aligned}
$$

Since α is the limit point of $\tilde{\mathbf{x}}^{K_{r'}}$, there exists $r'_0(\epsilon) > 0$ such that $\forall r' \geq r'_0(\epsilon)$,

$$
0 \leq \tilde{x}_j^{K_{r'}} < \tilde{\alpha}_j \quad \text{for } j \text{ such that } \alpha_j = 0\,, \tag{26}
$$

$$
|\tilde{x}_j^{K_{r'}} - \alpha_j| < \epsilon_1/2 \quad \text{for } j \text{ such that } \alpha_j > 0\,, \tag{27}
$$

and (27) implies that

$$
|\tilde{x}_j^{K_{r'}} - \tilde{\alpha}_j| < |\tilde{x}_j^{K_{r'}} - \alpha_j| + |\tilde{\alpha}_j - \alpha_j| < \epsilon_1, \quad \text{for } j \text{ such that } \alpha_j > 0\,.
$$

Step 2: Lower bound the demand-drop probability. Denote the average number of demands going from j' to k arrived per time unit during the first s time slots as $f_{j'k}[s]$. For stationary policy U, denote the average fraction of demand arriving at j' that is served by supply at i during this period as $d_{ij'}^U[s]$ (we omit the superscript U in the following for notational simplicity). For $t = 0, \frac{1}{K_{r'}}, \frac{2}{K_{r'}}, \cdots$, if $\bar{\mathbf{X}}^{K_{r'},U}[0] = \tilde{\mathbf{x}}^{K_{r'}}$ and no demand is dropped prior to $K_{r'}t$ we have

$$\bar{X}_i^{K_{r'},U}[t] - \tilde{x}_i^{K_{r'}} = \frac{K_{r'}t}{K_{r'}} \left(\sum_{j' \in V_D} f_{j'i}[K_{r'}t] - \sum_{j' \in \partial(i)} d_{ij'}[K_{r'}t] \left(\sum_{k \in V_S} f_{j'k}[K_{r'}t] \right) \right).$$

Since $\tilde{\alpha}_j > 0$, $\forall j \in V_S$, the Lyapunov function $L_{\tilde{\alpha}}(\cdot)$ is well-defined. Evaluate the Lyapunov function at $\left(\bar{\mathbf{X}}^{K_{r'},U}[t] - \tilde{\mathbf{x}}^{K_{r'}} + \tilde{\alpha} \right)$, we have:

$$\begin{aligned} & L_{\tilde{\alpha}} \left(\bar{\mathbf{X}}^{K_{r'},U}[t] - \tilde{\mathbf{x}}^{K_{r'}} + \tilde{\alpha} \right) & (28) \\ = & L_{\tilde{\alpha}} \left(t \left(\sum_{j' \in V_D} f_{j'i}[K_{r'}t] - \sum_{j' \in \partial(i)} d_{ij'}[K_{r'}t] \left(\sum_{k \in V_S} f_{j'k}[K_{r'}t] \right) \right)_{i \in V_S} + \tilde{\alpha} \right) \\ \overset{(a)}{=} & t L_{\tilde{\alpha}} \left(\left(\sum_{j' \in V_D} f_{j'i}[K_{r'}t] - \sum_{j' \in \partial(i)} d_{ij'}[K_{r'}t] \left(\sum_{k \in V_S} f_{j'k}[K_{r'}t] \right) \right)_{i \in V_S} + \tilde{\alpha} \right) \\ \geq & t \min_{\Delta \mathbf{x} \in \mathcal{X}_{\mathbf{f}}} L_{\tilde{\alpha}}(\tilde{\alpha} + \Delta \mathbf{x}). & (29) \end{aligned}$$

Equality (a) holds because the Lyapunov function is scale-invariant with respect to $\tilde{\alpha}$. Here $\Delta\mathbf{x}$ is the change of (normalized) state in $K_{r'}$ time slots given average demand arrival rate during this period $\mathbf{f}$, and $\mathcal{X}_{\mathbf{f}}$ is defined in (14).

Define $v_{\tilde{\alpha}}(\mathbf{f}) \triangleq \min_{\Delta\mathbf{x} \in \mathcal{X}_{\mathbf{f}}} L_{\tilde{\alpha}}(\tilde{\alpha} + \Delta\mathbf{x})$, which is the minimum rate the Lyapunov function increases under any policy, given current demand arrival rate $\mathbf{f}$. Now we construct a set of demand sample paths that must lead to demand drop before returning to the starting state. First note that $\{\mathbf{f} : v_{\tilde{\alpha}}(\mathbf{f}) > 0\}$ is non-empty. To see this, let $f'_{j'k}$ equal to 1 for some j' and $k \notin \partial(j')$, and 0 otherwise (such a pair (j', k) exists by Assumption 1). This $\mathbf{f}'$ results in a strictly positive $v_{\tilde{\alpha}}(\mathbf{f}')$. Therefore for any $\epsilon_2 > 0$ there exists demand arrival rate $\tilde{\mathbf{f}}$ such that

$$v_{\tilde{\alpha}}(\tilde{\mathbf{f}}) > 0 \qquad \text{and} \qquad \frac{\Lambda^*(\tilde{\mathbf{f}})}{v_{\tilde{\alpha}}(\tilde{\mathbf{f}})} \leq \inf_{\mathbf{f}: v_{\tilde{\alpha}}(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{v_{\tilde{\alpha}}(\mathbf{f})} + \epsilon_2.$$

It is not hard to show that $v_{\tilde{\alpha}}(\mathbf{f})$ is continuous in $\mathbf{f}$, hence there exists $\epsilon_3 > 0$ such that for any $\hat{\mathbf{f}} : ||\hat{\mathbf{f}} - \tilde{\mathbf{f}}||_\infty < \epsilon_3$, we have

$$v_{\tilde{\alpha}}(\hat{\mathbf{f}}) > 0 \qquad \text{and} \qquad v_{\tilde{\alpha}}(\hat{\mathbf{f}}) > (1 - \epsilon_2) v_{\tilde{\alpha}}(\tilde{\mathbf{f}}).$$

Denote $T \triangleq \frac{1 + \frac{\epsilon_1}{\min_{j:\alpha_j > 0} \alpha_j}}{(1-\epsilon_2) v_{\tilde{\alpha}}(\tilde{\mathbf{f}})}$, define

$$\mathcal{B}_{\tilde{\alpha}} \triangleq \left\{ \bar{\mathbf{A}}[t] \in C\,[0, T] \,\middle|\, \sup_{t \in [0,T]} ||\bar{\mathbf{A}}[t] - t\tilde{\mathbf{f}}||_\infty \leq \epsilon_3 \right\}.$$

For any fluid scale demand arrival sample path $\bar{\mathbf{A}}[\cdot] \in \mathcal{B}_{\tilde{\alpha}}$, we will show that for $t = 1, 2, \cdots, \lfloor K_{r'}T \rfloor$, the followings are true: (i)normalized state $\bar{\mathbf{X}}^{K_{r'},U}[t]$ does not hit $\tilde{\mathbf{x}}^{K_{r'}}$ before any demand is dropped; (ii)at least one demand is dropped.

To prove (i), note that function $\tilde{L}_{\tilde{\alpha}}(\mathbf{x}) \triangleq L_{\tilde{\alpha}}\left(\mathbf{x}+\tilde{\alpha}-\tilde{\mathbf{x}}^{K_{r'}}\right) > 0$ for any state in Ω except $\tilde{\mathbf{x}}^{K_{r'}}$. By inequality (29), if no demand is dropped during the first $\lfloor K_{r'}t \rfloor$ time slots we have:

$$\tilde{L}_{\tilde{\alpha}}\left(\bar{\mathbf{X}}^{K_{r'},U}[t]\right) \geq tv\left(\frac{1}{t}\bar{\mathbf{A}}[t]\right) \geq t \min_{\bar{\mathbf{A}}[\cdot]\in\mathcal{B}} v\left(\frac{1}{t}\bar{\mathbf{A}}[t]\right) > t(1-\epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}}) > 0.$$

We prove (ii) by contradiction. Suppose no demand is dropped given (fluid scale) demand arrival sample path $\bar{\mathbf{A}}[\cdot] \in \mathcal{B}$, then

$$\tilde{L}_{\tilde{\alpha}}\left(\bar{\mathbf{X}}^{K_{r'},U}[T]\right) \geq T \min_{\bar{\mathbf{A}}[\cdot]\in\mathcal{B}} v\left(\frac{1}{T}\bar{\mathbf{A}}[T]\right) > \frac{1+\frac{\epsilon_1}{\min_{j:\alpha_j>0}\alpha_j}}{(1-\epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}})}(1-\epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}}) = 1+\frac{\epsilon_1}{\min_{j:\alpha_j>0}\alpha_j}.$$

Expand the expression of $\tilde{L}_{\tilde{\alpha}}\left(\bar{\mathbf{X}}^{K_{r'},U}[T]\right)$, we have

$$1-\min_j \frac{\bar{X}_j^{K_{r'},U}[T]+\left(\tilde{\alpha}_j-\tilde{x}_j^{K_{r'}}\right)}{\tilde{\alpha}_j} > 1+\frac{\epsilon_1}{\min_{j:\alpha_j>0}\alpha_j},$$

$$\begin{aligned}\min\left\{\min_{j:\alpha_j=0}\frac{\bar{X}_j^{K_{r'},U}[T]}{\tilde{\alpha}_j}, \min_{j:\alpha_j>0}\frac{\bar{X}_j^{K_{r'},U}[T]-\epsilon_1/2}{\tilde{\alpha}_j}\right\} &\leq \min_j \frac{\bar{X}_j^{K_{r'},U}[T]+\left(\tilde{\alpha}_j-\tilde{x}_j^{K_{r'}}\right)}{\tilde{\alpha}_j} \\ &< -\frac{\epsilon_1}{\min_{j:\alpha_j>0}\alpha_j}.\end{aligned} \tag{30}$$

Note that the first inequality in (30) holds because of (26) and (27). Inequality (30) implies that $\min_j \bar{X}_j^{K_{r'},U}[T] < 0$, which is impossible as queue lengths must be non-negative.

Step 3: Obtain the asymptotic lower bound. We use renewal-reward theorem (Ross 1996) to lower bound the demand-drop probability. Recall that for the $K_{r'}$-th system, we have $\pi^{K_{r'}}(\tilde{\mathbf{x}}^{K_{r'}}) \geq K_{r'}^{-n}$. Consider the regenerative process that restarts each time $\bar{\mathbf{X}}^{K_{r'},U}[t] = \tilde{\mathbf{x}}^{K_{r'}}$. Without loss of generality, let $\bar{\mathbf{X}}^{K_{r'},U}[0] = \tilde{\mathbf{x}}^{K_{r'}}$. Define $\tau \triangleq \inf_{t\geq 1/K_{r'}}\left\{\bar{\mathbf{X}}^{K_{r'},U}[t] = \tilde{\mathbf{x}}^{K_{r'}}\right\}$. Using the result from step 2, we have:

$$\begin{aligned}\mathbb{P}_{\mathrm{o}}^{K_{r'},U} &= \frac{\mathbb{E}\left[\#\{\text{demand drop during } [0,\tau]\}\right]}{\mathbb{E}[\tau]} \\ &= \pi^{K_{r'}}\left(\tilde{\mathbf{x}}^{K_{r'}}\right)\mathbb{E}\left[\#\{\text{demand drop during } [0,\tau]\}\right] \\ &\geq K_{r'}^{-n}\mathbb{P}\left(\#\{\text{demand drop during } [0,\tau]\} \geq 1\right) \\ &\geq K_{r'}^{-n}\mathbb{P}\left(\bar{\mathbf{A}}^{K_{r'}}[\cdot] \in \mathcal{B}_{\tilde{\alpha}}\right).\end{aligned}$$

Take asymptotic limit on both sides, we have:

$$\begin{aligned}\liminf_{r'\to\infty}\frac{1}{K_{r'}}\log\mathbb{P}_{\mathrm{o}}^{K_{r'},U} &\geq \liminf_{r'\to\infty}\frac{1}{K_{r'}}\log\mathbb{P}\left(\bar{\mathbf{A}}^{K_{r'}}[\cdot]\in\mathcal{B}_{\tilde{\alpha}}\right) \\ &\overset{(a)}{\geq} -\inf_{\bar{\mathbf{A}}[\cdot]\in\mathcal{B}_{\tilde{\alpha}}^{o}\cap\mathrm{AC}[0,T]}\int_0^T \Lambda^*\left(\frac{d}{dt}\bar{\mathbf{A}}(t)\right)dt \\ &\overset{(b)}{\geq} -T\Lambda^*(\tilde{\mathbf{f}}) \\ &= -\frac{1+\frac{\epsilon_1}{\min_{j:\alpha_j>0}\alpha_j}}{(1-\epsilon_2)v_{\tilde{\alpha}}(\tilde{\mathbf{f}})}\Lambda^*(\tilde{\mathbf{f}})\end{aligned}$$

$$\geq -\frac{1+\frac{\epsilon_1}{\min_{j:\alpha_j>0}\alpha_j}}{1-\epsilon_2}\left(\inf_{\mathbf{f}:v_{\tilde{\alpha}}(\mathbf{f})>0}\frac{\Lambda^*(\mathbf{f})}{v_{\tilde{\alpha}}(\mathbf{f})}+\epsilon_2\right).$$

Here (a) holds because of Mogulskii's Theorem (Fact 1), (b) holds because demand sample path $\bar{\mathbf{A}}[t]=t\tilde{\mathbf{f}}\in \mathrm{AC}[0,T]$ is a member of $\mathcal{B}_{\tilde{\alpha}}$. For any $\delta>0$, by choosing small enough $\epsilon_1(\delta),\epsilon_2(\delta)>0$, we have

$$-\liminf_{r'\to\infty}\frac{1}{K_{r'}}\log\mathbb{P}_{\mathrm{o}}^{K_{r'},U}\leq(1+\delta)(\gamma_{\mathrm{CB}}(\tilde{\alpha}(\delta))+\delta).$$

Here the choice of $\tilde{\alpha}$ depends on δ. To get rid of the multiplicative term $(1+\delta)$, it suffices to show that $\sup_{\alpha\in\mathrm{relint}(\Omega)}\gamma_{\mathrm{CB}}(\alpha)<\infty$. This can be proved by the following construction: let $\bar{\mathbf{A}}[t]=t\mathbf{f}'$ for $t\in[0,1]$ where $f_{j'k}=1$ for some $j'\in V_D$ and $k\notin\partial(j')$. It is easy to show that $\sup_{\alpha\in\mathrm{relint}(\Omega)}\gamma_{\mathrm{CB}}(\alpha)\leq\Lambda^*(\mathbf{f}')<\infty$. Therefore by choosing a small enough δ, we have

$$-\liminf_{r'\to\infty}\frac{1}{K_{r'}}\log\mathbb{P}_{\mathrm{o}}^{K_{r'},U}\leq\gamma_{\mathrm{CB}}(\tilde{\alpha}(\epsilon))+\epsilon.$$

By the definition of subsequence $\{K_r'\}$, we have

$$-\liminf_{K\to\infty}\frac{1}{K}\log\mathbb{P}_{\mathrm{o}}^{K,U}\leq\gamma_{\mathrm{CB}}(\tilde{\alpha}(\epsilon))+\epsilon.$$

As a result, for any $\epsilon>0$ there exists $\alpha\in\Omega$ such that $-\liminf_{K\to\infty}\frac{1}{K}\log\mathbb{P}_{\mathrm{o}}^{K,U}\leq\sup_{\alpha\in\mathrm{relint}(\Omega)}\gamma_{\mathrm{CB}}(\alpha)+\epsilon$, therefore $-\liminf_{K\to\infty}\frac{1}{K}\log\mathbb{P}_{\mathrm{o}}^{K,U}\leq\sup_{\alpha\in\mathrm{relint}(\Omega)}\gamma_{\mathrm{CB}}(\alpha)$. $\square$

D Sufficient Conditions for Exponent Optimality: Proof of Proposition 3

The proof of Proposition 3 consists of two parts. We first derive an achievability bound for policies that satisfy the negative drift property in Proposition 3; we then show it matches the converse bound in Lemma 2 if the steepest descent property in Proposition 3 is also satisfied.

D.1 An achievability bound

The following lemma is an adaptation of Theorem 5 and Proposition 7 in Venkataramanan and Lin (2013) to our setting. It gives the achievability bound of the exponent of the steady state demand-drop probability, for any policy such that the negative drift condition in Proposition 3 is met for $L_\alpha(\cdot)$ where $\alpha\in\mathrm{relint}(\Omega)$. The main technical difficulty comes from the fact that it characterizes the *steady state* of the system. The analysis uses Freidlin-Wentzell theory and follows from Stolyar (2003), Venkataramanan and Lin (2013).

Lemma 7 (Achievability bound). *For the system being considered, if policy U satisfies the negative drift condition in Proposition 3 for $L_\alpha(\cdot)$ where $\alpha\in\mathrm{relint}(\Omega)$, we have (the subscript "AB" stands for achievability bound)*

$$-\limsup_{K\to\infty}\frac{1}{K}\log\mathbb{P}_{\mathrm{p}}^{K,U}\geq\gamma_{\mathrm{AB}}(\alpha)\,. \tag{31}$$

Here for fixed[12] $T > 0$,

$$\gamma_{\text{AB}}(\alpha) \triangleq \inf_{v>0,\mathbf{f},\bar{\mathbf{A}},\bar{\mathbf{X}}^U} \frac{\Lambda^*(\mathbf{f})}{v},$$
$$\textit{where } (\bar{\mathbf{A}}, \bar{\mathbf{X}}^U) \textit{ is a FSP on } [0,T] \textit{ under } U \textit{ such that for some regular } t \in [0,T]$$
$$\frac{d}{dt}\bar{\mathbf{A}}[t] = \mathbf{f}\,, \quad L_\alpha(\bar{\mathbf{X}}^U[t]) < 1\,, \quad \frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t]) = v\,.$$

Before proving Lemma 7, we first prove two technical lemmas that give an upper bound of first time the Lyapunov function is close to 0.

Lemma 8. *Fix $\alpha \in \operatorname{relint}(\Omega)$. Suppose that for policy $U \in \mathcal{U}$ and any fluid limit under this policy $(\bar{\mathbf{A}}[\cdot], \bar{\mathbf{X}}^U[\cdot])$, there exists $\eta > 0$ such that $\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t]) < -\eta$ for any $\bar{\mathbf{X}}^U[t] \neq \alpha$ and any regular t. Then we have, for any fluid limit at any $t \geq 1/\eta$,*

$$\lim_{K\to\infty} \mathbb{E}\left(L_\alpha\left(\bar{\mathbf{X}}^{K,U}[t]\right)\right) = 0\,. \tag{32}$$

Proof of Lemma 8. Let $\{K_r\}$ be any subsequence of $\{K\}$, then by definition of fluid limits (Dai 1995) and using the Arzela-Ascoli theorem, almost surely there is a further subsequence $\{K_{r'}\}$ such that, $\bar{\mathbf{X}}^{K_{r'},U}[\cdot]$ converges uniformly on compact sets to a fluid limit $\bar{\mathbf{X}}^U[\cdot]$. We know that any fluid limit $\bar{\mathbf{X}}^U[\cdot]$ is absolutely continuous, and it satisfies $\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t]) \leq -\eta$ for $\bar{\mathbf{X}}^U[t] \neq \alpha$ at any regular t. Since $L_\alpha(\mathbf{x}) \leq 1$ for any $\mathbf{x} \in \Omega$, we have $\bar{\mathbf{X}}^U[t] = 0$ for $t \geq 1/\eta$, hence $\lim_{r'\to\infty} L_\alpha\left(\bar{\mathbf{X}}^{K_{r'},U}[t]\right) = 0$ a.s. for $t \geq 1/\eta$. Because of the boundedness of $L_\alpha\left(\bar{\mathbf{X}}^{K_{r'},U}[t]\right)$, the sequence is uniformly integrable hence we have $\lim_{r'\to\infty} \mathbb{E}\left(L_\alpha\left(\bar{\mathbf{X}}^{K_{r'},U}[t]\right)\right) = 0$, for any $t \geq 1/\eta$.

We have shown that for any subsequence $\{K_r\}$ of $\{K\}$, there is a further subsequence $\{K_{r'}\}$ such that (32) holds, then (32) must hold for $\{K\}$. $\square$

We have the following lemma which bounds the expected time for $L_\alpha(\bar{\mathbf{X}}^{K,U}[t])$ to reach the level set $\{\mathbf{x} : L_\alpha(\mathbf{x}) \leq \epsilon\}$ for any $\epsilon \in (0,1)$.

Lemma 9. *Fix any $\epsilon > 0$. Under the same negative drift condition on policy U as in Lemma 8, define*

$$\beta_\epsilon^{K,U} \triangleq \inf\left\{t \geq 0 : L_\alpha(\bar{\mathbf{X}}^{K,U}[t]) \leq \epsilon\right\},$$

then there exists $K_0(\epsilon) < \infty$ such that for any $K > K_0(\epsilon)$ and any $\bar{\mathbf{X}}^{K,U}[0] \in \Omega$, we have

$$\mathbb{E}\left(\beta_\epsilon^{K,U} | \bar{\mathbf{X}}^{K,U}[0]\right) \leq \frac{2}{\epsilon\eta}(1+\epsilon)\,.$$

Proof of Lemma 9. We will first show that the K-th system satisfy a certain negative drift property for large enough K, then we apply Theorem 2.1(ii) in Meyn and Tweedie (1994) to obtain the result.

[12] The definition of quantity $\gamma_{\text{AB}}(\alpha)$ is based on the local behavior of $\bar{\mathbf{A}}$ and $\bar{\mathbf{X}}^U$ for times close to t. In particular, the value of T plays no role.

It follows from Lemma 8 that for each $t \geq 1/\eta$, $\epsilon_1 > 0$, $\exists K_1(\epsilon_1) < \infty$ such that for any $K > K_1(\epsilon_1)$:

$$\mathbb{E}\left(L_\alpha\left(\bar{\mathbf{X}}^{K,U}[t]\right)\right) \leq \epsilon_1 .$$

Define the "petite set" B as in Meyn and Tweedie (1994):

$$B \triangleq \{\mathbf{x} \in \Omega : L_\alpha(\mathbf{x}) \leq \epsilon\}, \tag{33}$$

then for $K > K_1(\epsilon/2)$,

$$\begin{aligned}&\mathbb{E}\left(L_\alpha\left(\bar{\mathbf{X}}^{K,U}[1/\eta]\right)\middle|\bar{\mathbf{X}}^{K,U}[0]\right)\\ \leq\ &\frac{\epsilon}{2}\\ \leq\ &L_\alpha\left(\bar{\mathbf{X}}^{K,U}[0]\right) - \frac{\epsilon}{2} + \epsilon\mathbb{I}\left\{\bar{\mathbf{X}}^{K,U}[0] \in B\right\} .\end{aligned}$$

Which can be interpreted as: except for the case where the initial Lyapunov function is almost 0, i.e. $\bar{\mathbf{X}}^{K,U}[0] \in B$, the expectation of Lyapunov function is guaranteed to decrease by at least $\epsilon/2$ after $1/\eta$ scaled time.

Apply Theorem 2.1(ii) of Meyn and Tweedie (1994), for each $K > K_1(\epsilon/2)$ and $\bar{\mathbf{X}}^{K,U}[0] \in \Omega$ we have

$$\mathbb{E}\left(\beta_\epsilon^{K,U}\middle|\bar{\mathbf{X}}^{K,U}[0]\right) \leq \frac{2}{\epsilon\eta}\left(L_\alpha\left(\bar{\mathbf{X}}^{K,U}[0]\right) + \epsilon\right) \leq \frac{2}{\epsilon\eta}(1+\epsilon) .$$

□

Proof of Lemma 7. *Step 1. Define stopping times and consider the sampling chain.* In this step, we mostly follow the approach in Venkataramanan and Lin (2013) (Freidlin-Wentzell theory) and decompose the expression for the likelihood of the Lyapunov function taking on a large value. There are minor differences between our proof and proof of Theorem 4 in Venkataramanan and Lin (2013) because of our closed queueing network setting, so we will write down each step for completeness.

In the following, let $\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}$ be a random vector distributed as the stationary distribution of recurrent class associated with initial (normalized) state $\mathbf{z} \in \Omega$. We want to upper bound:

$$\limsup_{K\to\infty} \frac{1}{K} \log\left(\max_{\mathbf{z}\in\Omega} \mathbb{P}\left(L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}) \geq 1\right)\right) .$$

Choose positive constants δ, ϵ such that $0 < \delta < \epsilon < 1$. Consider the following stopping times defined on a sample path $\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[\cdot]$:

$$\begin{aligned}\beta_1^{K,U} &\triangleq \inf\{t \geq 0 : L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[t]) \leq \delta\},\\ \eta_i^{K,U} &\triangleq \inf\{t \geq \beta_i^{K,U} : L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[t]) \geq \epsilon\}, \quad i = 1, 2, \cdots\\ \beta_i^{K,U} &\triangleq \inf\{t \geq \eta_{i-1}^{K,U} : L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[t]) \leq \delta\}, \quad i = 2, 3, \cdots\end{aligned}$$

Let the Markov chain $\hat{\mathbf{X}}_{\mathbf{z}}^{K,U}[i]$ be obtained by sampling $\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[t]$ at the stopping times $\eta_i^{K,U}$. Since $\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}$ is stationary, there must also exist a stationary distribution for Markov chain $\hat{\mathbf{X}}_{\mathbf{z}}^{K,U}[i]$. Let $\Theta_{\mathbf{z}}^{K,U}$ denote the state space of the sampled chain $\hat{\mathbf{X}}_{\mathbf{z}}^{K,U}[i]$, $\hat{\pi}_{\mathbf{z}}^{K,U}$ is the chain's stationary distribution.

The above construction was based on the following idea: first divide time into cycles, where the i-th cycle is the interval of time between consecutive η_i's, i.e., a cycle is completed each time the value of $L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U})$ goes down below δ and then rises above ϵ. Then the fraction of time the Lyapunov function spent above 1 is equal to the ratio $\mathbb{E}$[time for which $L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}) \geq 1$ during a cycle]/($\mathbb{E}$[length of cycle]) in steady state. Since we sample the initial state as $\mathbf{x} \sim \hat{\pi}_{\mathbf{z}}^{K,U}$, the first cycle itself characterizes the steady state ratio. Therefore, the stationary likelihood of event $\{L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}) \geq 1\}$ can be expressed as (see Lemma 10.1 in Stolyar 2003):

$$\mathbb{P}\left(L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}) \geq 1\right) = \frac{\int_{\Theta_{\mathbf{z}}^{K,U}} \hat{\pi}_{\mathbf{z}}^{K,U}(d\mathbf{x})\mathbb{E}\left(\int_0^{\eta_1^{K,U}} \mathbb{I}\left\{L_\alpha(\bar{\mathbf{X}}^{K,U}[t]) \geq 1\right\} dt \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)}{\int_{\Theta_{\mathbf{z}}^{K,U}} \hat{\pi}_{\mathbf{z}}^{K,U}(d\mathbf{x})\mathbb{E}(\eta_1^{K,U}|\bar{\mathbf{X}}^{K,U}[0] = \mathbf{x})} \,. \tag{34}$$

Step 2. Bounding the RHS of (34). To upper bound $\mathbb{P}\left(L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}) \geq 1\right)$, we lower bound the denominator in the RHS of (34) and upper bound the numerator.

- *Step 2a. Bounding the Denominator.*

  Since exactly one demand arrives per time period, we have

$$||\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[t+1] - \bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[t]||_\infty \leq 1 \,. \tag{35}$$

  Using property 2 of $L_\alpha(\mathbf{x})$ in Lemma 6, we further have

$$L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[\eta_1^{K,U}]) - L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[\beta_1^{K,U}]) \leq \frac{1}{\min_i \alpha_i}(\eta_1^{K,U} - \beta_1^{K,U}) \,,$$

$$L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[\eta_1^{K,U}]) \geq \epsilon - \frac{1}{\min_i \alpha_i}\frac{1}{K} \,, \qquad L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[\beta_1^{K,U}]) \leq \delta + \frac{1}{\min_i \alpha_i}\frac{1}{K} \,. \tag{36}$$

  Hence there exists $K_1 = K_1(\epsilon, \delta) > 0$ such that for any $K > K_1$, the denominator of (34) satisfies

$$\eta_1^{K,U} \geq \frac{\min_i \alpha_i}{2}(\epsilon - \delta) \,. \tag{37}$$

- *Step 2b. Bounding the Numerator.* This part is more complex, and we first decompose the numerator into several terms. Let $\rho \in (\epsilon, 1)$. Since $\epsilon < \rho$, by inequality (35) and Lipschitz continuity of $L_\alpha(\mathbf{x})$ there exists $K_2 = K_2(\epsilon, \rho) > 0$, such that $L(\bar{\mathbf{X}}^{K,U}[\eta_i^{K,U}]) \leq \rho$ for all $K \geq K_2$.

  We define another stopping time:

$$\eta^{K,U,\uparrow} \triangleq \inf\{t \geq 0 : L_\alpha(\bar{\mathbf{X}}^{K,U}[t]) \geq 1\} \,.$$

  Then for any $\mathbf{x} \in \Theta_{\mathbf{z}}^{K,U}$, we must have:

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{\eta_1^{K,U}} \mathbb{I}\{L_\alpha(\bar{\mathbf{X}}^{K,U}[t]) \geq 1\} dt \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right) \\
&\leq \mathbb{E}\left(\mathbb{I}\{\eta^{K,U,\uparrow} \leq \beta_1^{K,U}\}(\beta_1^{K,U} - \eta^{K,U,\uparrow}) \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right) \,.
\end{aligned}$$

The above inequality holds because:

- if $\beta_1^{K,U} \leq \eta^{K,U,\uparrow}$, then both sides are zero (because the Lyapunov function will hit ϵ before 1);
- if $\beta_1^{K,U} > \eta^{K,U,\uparrow}$, then $L_\alpha(\bar{\mathbf{X}}^{K,U}[t]) \geq 1$ can occur only for $t \in [\eta^{K,U,\uparrow}, \beta_1^{K,U}]$, and this time interval has length $\beta_1^{K,U} - \eta^{K,U,\uparrow}$.

Hence

$$\begin{aligned}&\mathbb{E}\left(\int_0^{\eta_1^{K,U}} \mathbb{I}\left\{L_\alpha(\bar{\mathbf{X}}^{K,U}[t]) \geq 1\right\} dt \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\\ &\leq \mathbb{E}\left(\beta_1^{K,U} - \eta^{K,U,\uparrow} \,\middle|\, \eta^{K,U,\uparrow} \leq \beta_1^{K,U}, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right) \mathbb{P}\left(\eta^{K,U,\uparrow} \leq \beta_1^{K,U} \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right).\end{aligned}$$

Define

$$\beta^{K,U}(\mathbf{x}) \triangleq \inf\left\{t \geq 0 : L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[t]) \leq \delta \,\middle|\, \bar{\mathbf{X}}_{\mathbf{z}}^{K,U}[0] = \mathbf{x}\right\}.$$

Using the properties of Markov chains and conditional expectation, we have:

$$\begin{aligned}&\mathbb{E}\left(\beta_1^{K,U} - \eta^{K,U,\uparrow} \,\middle|\, \eta^{K,U,\uparrow} \leq \beta_1^{K,U}, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\\ &= \mathbb{E}\left(\mathbb{E}\left(\beta^{K,U}\left(\bar{\mathbf{X}}^{K,U}(\eta^{K,U,\uparrow})\right)\right) \,\middle|\, \eta^{K,U,\uparrow} \leq \beta_1^{K,U}, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\\ &\leq \sup_{\mathbf{x}\in\Omega} \mathbb{E}\left(\beta_1^{K,U} \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right).\end{aligned}$$

Let T be a positive number which will be chosen later. Recall that $L_\alpha(\mathbf{x}) \leq \rho$ for all $\mathbf{x} \in \Theta_{\mathbf{z}}^{K,U}$ when $K \geq K_2$. Hence, for any such $\mathbf{x} \in \Theta_{\mathbf{z}}^{K,U}$, we have,

$$\begin{aligned}&\mathbb{E}\left(\int_0^{\eta_1^{K,U}} \mathbb{I}\left\{L_\alpha(\bar{\mathbf{X}}^{K,U}[t]) \geq 1\right\} dt \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\\ &\leq \mathbb{E}\left(\beta_1^{K,U} - \eta^{K,U,\uparrow} \,\middle|\, \eta^{K,U,\uparrow} \leq \beta_1^{K,U}, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right) \mathbb{P}\left(\eta^{K,U,\uparrow} \leq \beta_1^{K,U} \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\\ &\leq \left(\sup_{\mathbf{x}\in\Omega} \mathbb{E}\left(\beta_1^{K,U} \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\right)\left[\mathbb{P}\left(\eta^{K,U,\uparrow} \leq T \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\right.\\ &\qquad \left. + \mathbb{P}\left(\beta_1^{K,U} \geq T \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\right] \qquad \left(\text{using } \eta^{K,U,\uparrow} \leq \beta_1^{K,U} \;\Rightarrow\; \eta^{K,U,\uparrow} \leq T \text{ or } T \leq \beta_1^{K,U}\right)\\ &\leq \underbrace{\left(\sup_{\mathbf{x}\in\Omega} \mathbb{E}\left(\beta_1^{K,U} \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)\right)}_{\text{(a)}}\left[\underbrace{\sup_{\mathbf{x}:L_\alpha(\mathbf{x})\leq\rho} \mathbb{P}\left(\eta^{K,U,\uparrow} \leq T \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)}_{\text{(b)}}\right.\\ &\qquad \left. + \underbrace{\sup_{\mathbf{x}:L_\alpha(\mathbf{x})\leq\rho} \mathbb{P}\left(\beta_1^{K,U} \geq T \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x}\right)}_{\text{(c)}}\right]. \end{aligned} \tag{38}$$

Note that all the terms are independent of $\mathbf{z}$.

— *Step 2b(i). Bounding term (a).* Apply Lemma 9, there exists $K_3 = K_3(\delta)$ such that for $K \geq K_3$, we have $(a) \leq 2(1+\delta)/(\eta\delta)$.

— *Step 2b(ii). Asymptotics for (b).* For $K \to \infty$, apply Proposition 2 in Venkataramanan and Lin (2013) to $\bar{\mathbf{X}}^{K,U}[\cdot]$, which doesn't require $\bar{\mathbf{X}}^{K,U}[\cdot]$ to be irreducible. We have:

$$\begin{aligned}
&\limsup_{K\to\infty} \frac{1}{K} \log \left(\sup_{\mathbf{x}: L_\alpha(\mathbf{x}) \leq \rho} \mathbb{P}\left(\eta^{K,U,\uparrow} \leq T \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x} \right) \right) \\
\leq \; & -\inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}^U} \int_0^T \Lambda^* \left(\frac{d}{dt} \bar{\mathbf{A}}[t] \right) dt, \; \textit{where } (\bar{\mathbf{A}}, \bar{\mathbf{X}}^U) \textit{ is an FSP} \\
& \textit{such that } L_\alpha(\bar{\mathbf{X}}^U[0]) \leq \rho, \; L_\alpha(\bar{\mathbf{X}}^U[t]) \geq 1 \textit{ for some } t \in [0,T] \,.
\end{aligned}$$

— *Step 2b(iii). Asymptotics for (c).* Intuitively, term (c) is the tail probability of the duration of a cycle that terminates when the Lyapunov function hit ρ. It remains to be shown that this term is negligible comparing to (b) as $T \to \infty$.
Proceed exactly the same as in proof of Theorem 4, part b(3) in Venkataramanan and Lin (2013), which only uses the condition (2) in Proposition 3. We obtain:

$$\limsup_{K\to\infty} \frac{1}{K} \log \left(\sup_{\mathbf{x}: L_\alpha(\mathbf{x}) \leq \rho} \mathbb{P}\left(\beta_1^{K,U} \geq T \,\middle|\, \bar{\mathbf{X}}^{K,U}[0] = \mathbf{x} \right) \right) \leq -\frac{T\eta + 2(\delta - \rho)}{2/\min_i \alpha_i + \eta} J_{\min} \,,$$

where $J_{\min} = \min_{\mathbf{f} \notin B(\phi, \epsilon')} \Lambda^*(\mathbf{f}) > 0$ and ϵ' is the ϵ specified in condition (2) of Proposition 3. It is not hard to see that as $T \to \infty$, RHS can be made arbitrarily small.

Now combine all the terms. For fixed ϵ, δ, ρ, note that the denominator of (34) and (a) in (38) are bounded by a constant term, so they have no contribution to the exponent of (34). Since as $T \to \infty$, (c) in (38) have an arbitrarily small exponent, we have

$$\begin{aligned}
&\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}_{\mathrm{p}}^{K,U} \\
\leq \; & \limsup_{K\to\infty} \frac{1}{K} \log \left(\max_{\mathbf{z} \in \Omega} \mathbb{P}\left(L_\alpha(\bar{\mathbf{X}}_{\mathbf{z}}^{K,U}) \geq 1 \right) \right) \qquad (39) \\
\leq \; & -\inf_{T>0} \inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}^U} \int_0^T \Lambda^* \left(\frac{d}{dt} \bar{\mathbf{A}}[t] \right) dt \\
& \textit{where } \bar{\mathbf{A}}, \bar{\mathbf{X}}^U \textit{ is an FSP such that } L_\alpha(\bar{\mathbf{X}}[0]) = \rho, \; L_\alpha(\bar{\mathbf{X}}[T]) \geq 1 \,. \qquad (40)
\end{aligned}$$

Finally, let $\delta, \epsilon, \rho \to 0$, we have

$$\begin{aligned}
&\limsup_{K\to\infty} \frac{1}{K} \log \mathbb{P}_{\mathrm{p}}^{K,U} \\
\leq \; & -\inf_{T>0} \inf_{\bar{\mathbf{A}}, \bar{\mathbf{X}}^U} \int_0^T \Lambda^* \left(\frac{d}{dt} \bar{\mathbf{A}}[t] \right) dt \\
& \textit{where } \bar{\mathbf{A}}, \bar{\mathbf{X}}^U \textit{ is an FSP such that } L_\alpha(\bar{\mathbf{X}}[0]) = 0, \; L_\alpha(\bar{\mathbf{X}}[T]) \geq 1 \,.
\end{aligned}$$

We briefly summarize Step 2 and provide some intuition. The goal is to upper bound the stationary likelihood that the Lyapunov function equals 1. To study the stationary behavior, we first divide time into cycles, where a cycle is completed each time the Lyapunov function goes down below δ then rises above ϵ, where $\delta < \epsilon \ll 1$. Then using a variant of renewal-reward theorem (equation (34)), we only need to lower bound the expected cycle duration, and upper bound the expected time the Lyapunov function stays at 1 during a cycle. The Lipschitz property of demand sample paths and the Lyapunov function ensures that the cycle duration is bounded away from 0 hence has no contribution to the *exponent* of the desired likelihood (Lemma 6). Meanwhile, Lemma 9 upper bounds the expected time until the Lyapunov function returns to δ after hitting 1. This leaves the exponent of the desired likelihood to be solely dependent on the probability that the Lyapunov function ever hit 1 during a cycle. Finally we apply the sample path large deviation principle (Fact 1) to bound this quantity.

Step 3. Reduce (39) to an one-dimensional variational problem. This rest of the proof is exactly the same as the proof of Theorem 5 and Proposition 7 in Venkataramanan and Lin (2013); we provide the intuition and omit the details.

The proof up until this point dealt with the *steady state* of the system. Recall the link between the exponent and value of a differential game described in Section 5.3. We now lower bound the exponent of the steady state demand drop probability by a variational problem (differential game), namely, (40). Since we are trying to lower bound the adversary's cost, we consider an "ideal adversary" who can increase $L_\alpha(\mathbf{x})$ at the minimum cost at *each* level set. Mathematically,

$$\text{The quantity in (40)} \leq -\inf_{T>0} \theta_T\,, \tag{41}$$

where

$$\begin{aligned}\theta_T \triangleq\ & \inf_{L[\cdot]} \int_0^T l_{\alpha,T}\left(L[t], \frac{d}{dt}L[t]\right) dt \\ & \text{s.t. } L[\cdot] \text{ is absolutely continuous and } L[0]=0, \quad L[T]\geq 1\,.\end{aligned}$$

$$\begin{aligned}l_{\alpha,T}(y,v) \triangleq\ & \inf_{\bar{\mathbf{A}},\bar{\mathbf{X}}^U} \Lambda^*(\mathbf{f}) \\ & \text{s.t. } (\bar{\mathbf{A}},\bar{\mathbf{X}}^U) \text{ is an FSP on } [0,T] \text{ such that for some regular } t\in[0,T] \\ & \quad \frac{d}{dt}\bar{\mathbf{A}}[t]=\mathbf{f}, \quad L_\alpha(\bar{\mathbf{X}}^U[t])=y, \quad \frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t])=v\,.\end{aligned}$$

Using the scale-invariance property of $L_\alpha(\mathbf{x})$ (Lemma 1), we can show that $l_{\alpha,T}(y,v)$ is independent of y (Proposition 7 in Venkataramanan and Lin 2013). As a result, the above variational problem reduces to an one-dimensional problem where the "ideal adversary" chooses a single rate (i.e., v in the statement of Lemma 7) at which $L_\alpha(\mathbf{x})$ increases. This problem is exactly the one in the statement of Lemma 7.

□

D.2 Converse Bound Matches Achievability Bound

In Lemma 2 we obtain the converse bound of any state-dependent policy. However, for a given policy U it's not obvious which α to plug in on the RHS of (15). In the following Lemma, we show that for policies that satisfy the negative drift property in Proposition 3 for Lyapunov function $L_\alpha(\cdot)$ where $\alpha\in\mathrm{relint}(\Omega)$, the converse bound is given by $\gamma_{\text{CB}}(\alpha)$.

Lemma 10. *For policies $U \in \mathcal{U}$ that satisfy the negative drift condition in the statement of Proposition 3 for $\alpha \in \operatorname{relint}(\Omega)$, we have*

$$-\liminf_{K\to\infty} \frac{1}{K} \log P_{\mathrm{o}}^{K,U} \leq \gamma_{\mathrm{CB}}(\alpha)\,.$$

Proof. The following proof is very similar to the proof of Lemma 2. We will emphasize the parts that are different and skip the repetitive arguments.

Step 1: Show that α is the 'resting point' of U. Using Lemma 8, for $\epsilon_1 > 0$, there exists $K_0 = K_0(\epsilon_1) > 0$ such that

$$\mathbb{E}\left(L_\alpha \left(\bar{\mathbf{X}}^{K,U}\left[1/\eta + T\right]\right) \middle| \bar{\mathbf{X}}^{K,U}[T] \right) \leq \epsilon_1\,, \qquad \forall K > K_0 \text{ and } T \geq 0\,.$$

By Markov inequality, we have for $K > K_0$, $\mathbb{P}\left(L_\alpha \left(\bar{\mathbf{X}}^{K,U}\left[1/\eta + T\right]\right) \geq 2\epsilon_1 \middle| \bar{\mathbf{X}}^{K,U}[T] \right) \leq \frac{1}{2}$, $\forall K > K_0$ and $T \geq 0$. In other words, starting from any time, with probability at least 1/2 the system state will be close to α after K/η periods.

Step 2: Lower bound the demand-drop probability. Proceed exactly as Step 2 and Step 3 in the proof of Lemma 2, we explicitly construct a demand sample path that guarantees a demand drop within any consecutive $\Theta(K)$ periods given the starting state satisfies $L_\alpha \left(\bar{\mathbf{X}}^{K,U}\left[1/\eta + T\right]\right) < 2\epsilon_1$. Then we obtain the desired result.

$\square$

Now we combine Lemma 7 and Lemma 10 to prove Proposition 3. Note that Lemma 1 and the steepest descent property in Proposition 3 are crucial in showing $\gamma_{\mathrm{AB}}(\alpha) \geq \gamma_{\mathrm{CB}}(\alpha)$ (the other direction is obvious).

Proof of Proposition 3.

Proof of Proposition 3. Let $U \in \mathcal{U}$ satisfy the conditions in Proposition 3. Then for regular t we have

$$\begin{aligned}
\frac{d}{dt} L_\alpha(\bar{\mathbf{X}}^U[t]) &\leq \inf_{U' \in \mathcal{U}} \left\{ \frac{d}{dt} L_\alpha(\bar{\mathbf{X}}^{U'}[t]) \middle| \frac{d}{dt}\bar{\mathbf{A}}'[t] = \mathbf{f} \right\} && \text{(steepest descent)} \\
&= \min_{\Delta\mathbf{x}\in\mathcal{X}_{\mathbf{f}}} \lim_{\Delta t\to 0} \frac{L_\alpha(\bar{\mathbf{X}}^{U'}[t] + \Delta\mathbf{x}\Delta t) - L_\alpha(\bar{\mathbf{X}}^{U'}[t])}{\Delta t} \\
&\leq \min_{\Delta\mathbf{x}\in\mathcal{X}_{\mathbf{f}}} \lim_{\Delta t\to 0} \frac{L_\alpha(\alpha + \Delta\mathbf{x}\Delta t)}{\Delta t} && \text{(sub-additivity, Lemma 1)} \\
&= \min_{\Delta\mathbf{x}\in\mathcal{X}_{\mathbf{f}}} L_\alpha(\alpha + \Delta\mathbf{x}) = v_\alpha(\mathbf{f})\,. && \text{(scale-invariance, Lemma 1)}
\end{aligned}$$

Let $v = \frac{d}{dt} L_\alpha(\bar{\mathbf{X}}^U[t])$, from $v \leq v_\alpha(\mathbf{f})$ we have $\{v > 0\} \subset \{v_\alpha(\mathbf{f}) > 0\}$, hence using Lemma 7 we have

$$\gamma_{\mathrm{AB}}(\alpha) = \inf_{v>0,\mathbf{f},\bar{\mathbf{A}},\bar{\mathbf{X}}^U} \frac{\Lambda^*(\mathbf{f})}{v} \geq \inf_{\mathbf{f}:v_\alpha(\mathbf{f})>0} \frac{\Lambda^*(\mathbf{f})}{v_\alpha(\mathbf{f})} = \gamma_{\mathrm{CB}}(\alpha)\,.$$

But since by Lemma 10 we know γ_{CB} is a converse bound for policy U, hence $\gamma_{\mathrm{AB}}(\alpha) \leq \gamma_{\mathrm{CB}}(\alpha)$. Therefore $\gamma_{\mathrm{AB}}(\alpha) = \gamma_{\mathrm{CB}}(\alpha)$. $\square$

E SMW Policies and Explicit Exponent

E.1 Lyapunov Drift of FSPs under SMW: Proof of Lemma 3

Proof. For notation simplicity, we will write $S_1(\bar{\mathbf{X}}^U[t])$ as S_1, $S_2\left(\bar{\mathbf{X}}^U[t], \frac{d}{dt}\bar{\mathbf{X}}^U[t]\right)$ as S_2, and $\min_{k\in S_1}\frac{d}{dt}\frac{\bar{\mathbf{X}}_k^U[t]}{\alpha_k}$ as c in the following. Let $(\bar{\mathbf{A}}, \bar{\mathbf{X}}^U)$ be an FSP under policy $U \in \mathcal{U}$.

- *Proof of (17).* Using the definition of derivatives, $\forall \delta > 0$, $\exists \epsilon_0 > 0$ such that $\forall \epsilon \in (0, \epsilon_0)$,

$$\frac{\bar{\mathbf{X}}_k^U(t+\epsilon)}{\alpha_k} - \frac{\bar{\mathbf{X}}_k^U(t)}{\alpha_k} \geq (c-\delta)\epsilon\,, \qquad \forall k \in S_1\,.$$

  Hence

$$\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t]) = -\lim_{\epsilon\to 0}\frac{\min_{k\in S_1}\frac{\bar{\mathbf{X}}_k^U(t+\epsilon)}{\alpha_k} - \min_{k\in S_1}\frac{\bar{\mathbf{X}}_k^U(t)}{\alpha_k}}{\epsilon} \leq -\lim_{\epsilon\to 0}\frac{(c-\delta)\epsilon}{\epsilon} = -c+\delta\,.$$

  Since $\delta > 0$ can be chosen arbitrarily, we have $\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t]) \leq -c$. Similarly, we can show that $\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t]) \geq -c$, hence $\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t]) = -c$.

- *Proof of (18).* Define $c' \triangleq \operatorname{argmin}_{k\in S_1\backslash S_2}\frac{d}{dt}\frac{\bar{\mathbf{X}}_k^U[t]}{\alpha_k}$, then $c' > c$. Using the definition of derivatives, there exists $\epsilon_0 > 0$ such that $\forall \epsilon \in [0, \epsilon_0]$,

$$\frac{\bar{\mathbf{X}}_k^U(t+\epsilon)}{\alpha_k} - \frac{\bar{\mathbf{X}}_k^U(t)}{\alpha_k} \geq \left(c' - \frac{c'-c}{3}\right)\epsilon, \qquad \forall k \in S_1\backslash S_2 \tag{42}$$

$$\frac{\bar{\mathbf{X}}_k^U(t+\epsilon)}{\alpha_k} - \frac{\bar{\mathbf{X}}_k^U(t)}{\alpha_k} \leq \left(c + \frac{c'-c}{3}\right)\epsilon, \qquad \forall k \in S_2\,. \tag{43}$$

  Since $(\bar{\mathbf{A}}, \bar{\mathbf{X}}^U)$ is an FSP, by definition there exists an increasing positive integer sequence $\{K_r\}$ such that $(\bar{\mathbf{A}}^{K_r}[\cdot], \bar{\mathbf{X}}^{K_r,U}[0], \Psi^{K_r,U}(\bar{\mathbf{A}}^{K_r}[\cdot], \bar{\mathbf{X}}^{K_r,U}[0]))$ converges uniformly to $(\bar{\mathbf{A}}, \bar{\mathbf{X}}^U)$. Using the definition of uniform convergence, for any $M > 1$ there exists r_0 such that for any $r > r_0$ and $t \in [0, T]$, we have $\forall k \in V_S$,

$$\left|\frac{\bar{\mathbf{X}}_k^{K_r,U}[t]}{\alpha_k} - \frac{\bar{\mathbf{X}}_k^U[t]}{\alpha_k}\right| < \frac{c'-c}{6}\frac{\epsilon_0}{M}\,. \tag{44}$$

  As a result, for any $\epsilon \in \left(\frac{\epsilon_0}{M}, \epsilon_0\right)$ and $r \geq r_0$, let $k \in S_1\backslash S_2$, $l \in S_2$, we have

$$\begin{aligned}
\frac{\bar{\mathbf{X}}_k^{K_r,U}(t+\epsilon)}{\alpha_k} &\geq \frac{\bar{\mathbf{X}}_k^U(t+\epsilon)}{\alpha_k} - \frac{c'-c}{6}\frac{\epsilon_0}{M} && \text{(plug in (44))}\\
&> \frac{\bar{\mathbf{X}}_k^U(t)}{\alpha_k} + \left(c' - \frac{c'-c}{3}\right)\epsilon - \frac{c'-c}{6}\epsilon && \text{(plug in (42))}\\
&\geq \frac{\bar{\mathbf{X}}_l^U(t+\epsilon)}{\alpha_l} + \frac{c'-c}{6}\epsilon && \text{(plug in (43))}\\
&\geq \frac{\bar{\mathbf{X}}_l^{K_r,U}(t+\epsilon)}{\alpha_l} && \text{(plug in (44)).}
\end{aligned}$$

  In other words, for any of the K_r-th system such that $r > r_0$, any queue in set S_2 has smaller scaled queue length than any queue in $V_S\backslash S_2$ during $(t + \frac{\epsilon_0}{M}, t + \epsilon_0)$. As a result, queues in S_2 alone determine the value of the Lyapunov function.

  By definition of SMW(α), the K_r-th system ($r > r_0$) will use the supplies within $V_S\backslash S_2$ to serve all demands arriving at $\partial(V_S\backslash S_2)$ during (scaled) time $(t + \frac{\epsilon_0}{M}, t + \epsilon_0)$. Moreover, queue lengths in $V_S \backslash S_2$ remain strictly positive during this period. Using the uniform convergence to FSP and the non-idling property of SMW policies, there exists $r_1 > 0$ such that for $r > r_1$,

$$\sum_{k \in V_S \backslash S_2} \left(\bar{\mathbf{X}}_k^{K_r, \text{SMW}(\alpha)}[t+\epsilon_0] - \bar{\mathbf{X}}_k^{K_r, \text{SMW}(\alpha)}\left[t + \frac{\epsilon_0}{M}\right] \right)$$
$$= \sum_{j' \in V_D, k \in V_S \backslash S_2} \left(\bar{\mathbf{A}}_{j'k}^{K_r}[t+\epsilon_0] - \bar{\mathbf{A}}_{j'k}^{K_r}\left[t + \frac{\epsilon_0}{M}\right] \right) - \sum_{j' \in \partial(V_S \backslash S_2), k \in V_S} \left(\bar{\mathbf{A}}_{j'k}^{K_r}[t+\epsilon_0] - \bar{\mathbf{A}}_{j'k}^{K_r}\left[t + \frac{\epsilon_0}{M}\right] \right).$$

Recall the Lipschitz property of $\bar{\mathbf{X}}^{K_r, \text{SMW}(\alpha)}[\cdot]$, we have

$$\left| \sum_{k \in V_S \backslash S_2} \left(\bar{\mathbf{X}}_k^{K_r, \text{SMW}(\alpha)}\left[t + \frac{\epsilon_0}{M}\right] - \bar{\mathbf{X}}_k^{K_r, \text{SMW}(\alpha)}[t] \right) \right| \leq \frac{\epsilon_0}{M}.$$

Combined, we have:

$$\begin{aligned}
&\sum_{k \in V_S \backslash S_2} \left(\bar{\mathbf{X}}_k^{K_r, \text{SMW}(\alpha)}[t+\epsilon_0] - \bar{\mathbf{X}}_k^{K_r, \text{SMW}(\alpha)}[t] \right) \\
\leq &\sum_{j' \in V_D, k \in V_S \backslash S_2} \left(\bar{\mathbf{A}}_{j'k}^{K_r}[t+\epsilon_0] - \bar{\mathbf{A}}_{j'k}^{K_r}\left[t + \frac{\epsilon_0}{M}\right] \right) \\
&- \sum_{j' \in \partial(V_S \backslash S_2), k \in V_S} \left(\bar{\mathbf{A}}_{j'k}^{K_r}[t+\epsilon_0] - \bar{\mathbf{A}}_{j'k}^{K_r}\left[t + \frac{\epsilon_0}{M}\right] \right) + \frac{\epsilon_0}{M}.
\end{aligned}$$

First let $r \to \infty$, then let $M \to \infty$. We have

$$\begin{aligned}
&\sum_{k \in V_S \backslash S_2} \left(\bar{\mathbf{X}}_k^{\text{SMW}(\alpha)}[t+\epsilon_0] - \bar{\mathbf{X}}_k^{\text{SMW}(\alpha)}[t] \right) \\
\leq &\sum_{j' \in V_D, k \in V_S \backslash S_2} \left(\bar{\mathbf{A}}_{j'k}[t+\epsilon_0] - \bar{\mathbf{A}}_{j'k}[t] \right) - \sum_{j' \in \partial(V_S \backslash S_2), k \in V_S} \left(\bar{\mathbf{A}}_{j'k}[t+\epsilon_0] - \bar{\mathbf{A}}_{j'k}[t] \right).
\end{aligned}$$

Since it is a closed system, we have:

$$\begin{aligned}
&\sum_{k \in S_2} \left(\bar{\mathbf{X}}_k^{\text{SMW}(\alpha)}[t+\epsilon_0] - \bar{\mathbf{X}}_k^{\text{SMW}(\alpha)}[t] \right) \\
= &- \sum_{k \in V_S \backslash S_2} \left(\bar{\mathbf{X}}_k^{\text{SMW}(\alpha)}[t+\epsilon_0] - \bar{\mathbf{X}}_k^{\text{SMW}(\alpha)}[t] \right) \\
\geq &\sum_{j' \in \partial(V_S \backslash S_2), k \in V_S} \left(\bar{\mathbf{A}}_{j'k}[t+\epsilon_0] - \bar{\mathbf{A}}_{j'k}[t] \right) - \sum_{j' \in V_D, k \in V_S \backslash S_2} \left(\bar{\mathbf{A}}_{j'k}[t+\epsilon_0] - \bar{\mathbf{A}}_{j'k}[t] \right).
\end{aligned}$$

Divide both sides by ϵ_0 and let $\epsilon_0 \to 0$. Since t is regular, we have:

$$\sum_{k \in S_2} \frac{d}{dt} \bar{\mathbf{X}}_k^{\text{SMW}(\alpha)}[t] \geq \left(\sum_{j' \in \partial(V_S \backslash S_2), k \in V_S} \frac{d}{dt} \bar{\mathbf{A}}_{j'k}[t] - \sum_{j' \in V_D, k \in V_S \backslash S_2} \frac{d}{dt} \bar{\mathbf{A}}_{j'k}[t] \right). \tag{45}$$

But notice that for any non-idling policy U', it cannot use the supplies in $V_S \backslash S_2$ to serve the demand arising out of $\partial(V_S \backslash S_2)$. Using the same analysis as above, for any non-idling policy

U' we have

$$\sum_{k\in S_2}\frac{d}{dt}\bar{\mathbf{X}}_k^{\mathrm{SMW}(\alpha)}[t]\le\left(\sum_{j'\in\partial(V_S\setminus S_2),k\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]-\sum_{j'\in V_D,k\in V_S\setminus S_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]\right). \tag{46}$$

Note that

$$\begin{aligned}&\sum_{j'\in\partial(V_S\setminus S_2),k\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]-\sum_{j'\in V_D,k\in V_S\setminus S_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]\\ =&\left(\sum_{j'\in V_D,k\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]-\sum_{j':\partial(j')\subset S_2,k\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]\right)\\ &-\left(\sum_{j'\in V_D,k\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]-\sum_{j'\in V_D,k\in S_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]\right)\\ =&\sum_{j'\in V_D,k\in S_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]-\sum_{j':\partial(j')\subset S_2,k\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]\,.\end{aligned}$$

Finally, observe that for any $k\in S_2$

$$\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}^U[t])=-\frac{d}{dt}\frac{\bar{\mathbf{X}}_k^U[t]}{\alpha_k}=-\frac{1}{\mathbf{1}_{S_2}^{\mathrm{T}}\alpha}\sum_{k\in S_2}\alpha_k\frac{d}{dt}\frac{\bar{\mathbf{X}}_k^U[t]}{\alpha_k}=-\frac{1}{\mathbf{1}_{S_2}^{\mathrm{T}}\alpha}\sum_{k\in S_2}\frac{d}{dt}\bar{\mathbf{X}}_k^U[t]\,. \tag{47}$$

Plug (45) and (46) into (47), we know that inequality (17) holds, and it becomes equality for SMW(α) policy.

□

E.2 Lyapunov Drift of Fluid Limits under SMW: Proof of Lemma 4

Proof of Lemma 4. **Negative drift.** Let $(\bar{\mathbf{A}},\bar{\mathbf{X}})$ be a fluid limit of the system under SMW(α), and t be its regular point. Simply plug in Lemma 3, and replace $\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]$ with $\phi_{j'k}$, we have (S_2 is defined in Lemma 3)

$$\begin{aligned}\frac{d}{dt}L_\alpha(\bar{\mathbf{X}}[t])=&-\frac{1}{\mathbf{1}_{S_2}^T\alpha}\left(\sum_{j'\in V_D,k\in S_2}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]-\sum_{j'\in V_D:\partial(j')\subseteq S_2,k\in V_S}\frac{d}{dt}\bar{\mathbf{A}}_{j'k}[t]\right)\\ \le&-\min_{S_2\subsetneq V_S}\frac{1}{\mathbf{1}_{S_2}^T\alpha}\left(\sum_{j'\in V_D,k\in S_2}\phi_{j'k}-\sum_{j'\in V_D:\partial(j')\subseteq S_2,k\in V_S}\phi_{j'k}\right)\\ \le&-\min_{S_2\subsetneq V_S}\left(\sum_{j'\in V_D,k\in S_2}\phi_{j'k}-\sum_{j'\in V_D:\partial(j')\subseteq S_2,k\in V_S}\phi_{j'k}\right)\\ \le&-\min\{\xi,\lambda_{\min}\}\,.\end{aligned}$$

Here $\lambda_{\min}\triangleq\min_{i\in V_S}\mathbf{1}^T\phi_{(i)}>0$ is the minimum supply arrival rate at any location (that has positive arrival rate), and $\xi\triangleq\min_{J\subsetneq V_D,J\neq\emptyset}\left(\sum_{i\in\partial(J)}\mathbf{1}^T\phi_{(i)}-\sum_{j'\in J}\mathbf{1}^T\phi_{j'}\right)>0$ is the Hall's gap of the system.

Robustness of drift. Define

$$G(\mathbf{f}) \triangleq \min_{S \subsetneq V_S} \left(\sum_{j' \in V_D, k \in A} \mathbf{f}_{j'k} - \sum_{j' : \partial(j') \subseteq S, k \in V_S} \mathbf{f}_{j'k} \right) .$$

Note that $G(\mathbf{f})$ is continuous in $\mathbf{f}$. Since $G(\phi) \le -\min\{\xi, \lambda_{\min}\} < 0$, by continuity there exists ϵ such that for any $\frac{d}{dt}\bar{\mathbf{A}}[t] \in B(\phi, \epsilon)$,

$$\frac{d}{dt} L_\alpha(\bar{\mathbf{X}}[t]) = G\left(\frac{d}{dt}\bar{\mathbf{A}}[t]\right) \le -\frac{1}{2}\min\{\xi, \lambda_{\min}\} .$$

□

E.3 Explicit Exponent and Most Likely Sample Path: Proof of Lemma 5

Proof of Lemma 5. **Explicit exponent.** Let $(\mathbf{A}[\cdot], \mathbf{X}[\cdot])$ be a fluid sample path under SMW(α). For a regular point t of this FSP, denote $\mathbf{f} \triangleq \frac{d}{dt}\bar{\mathbf{A}}[t]$. Since exactly one demand arrives per period, we have $\mathbf{f} \in \mathcal{F} \triangleq \{\mathbf{f} \in \mathbb{R}^{n\times n} : f_{j'k} \ge 0 \ \forall j' \in V_D, k \in V_S; \sum_{j' \in V_D} \sum_{k \in V_S} f_{j'k} = 1\}$.

For notation simplicity, for $S \subset V_S$ denote

$$\mathrm{gap}_S(\mathbf{f}) \triangleq \sum_{j' : \partial(j') \subseteq S, k \in V_S} \mathbf{f}_{j'k} - \sum_{j' \in V_D, k \in S} \mathbf{f}_{j'k} .$$

In words, $\mathrm{gap}_S(\mathbf{f})$ is the minimum net rate at which supply in S is drained given current demand arrival rate $\mathbf{f}$, assuming no demand is dropped. Using the result of Lemma 3, we have:

$$\frac{d}{dt} L_\alpha(\bar{\mathbf{X}}[t]) = \bar{v}(\mathbf{f}) \triangleq \max_{S \subseteq V_S} \frac{\mathrm{gap}_S(\mathbf{f})}{\mathbf{1}_S^T \alpha} . \tag{48}$$

Recall the definition of $\gamma_{\mathrm{AB}}(\alpha)$ in Lemma 7. We have

$$\begin{aligned}
\gamma_{\mathrm{AB}}(\alpha) &= \min_{\mathbf{f} \in \mathcal{F} : \bar{v}(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{\bar{v}(\mathbf{f})} \\
&= \min_{\mathbf{f} \in \mathcal{F} : \max_{S \subseteq V_S} \mathrm{gap}_S(\mathbf{f}) > 0} \frac{\Lambda^*(\mathbf{f})}{\max_{S \subseteq V_S} \frac{\mathrm{gap}_S(\mathbf{f})}{\mathbf{1}_S^T \alpha}} \\
&= \min_{\mathbf{f} \in \mathcal{F} : \max_{S \subseteq V_S} \mathrm{gap}_S(\mathbf{f}) > 0} \left\{ \min_{S \subseteq V_S : \mathrm{gap}_S(\mathbf{f}) > 0} \left(\mathbf{1}_S^T \alpha\right) \frac{\Lambda^*(\mathbf{f})}{\mathrm{gap}_S(\mathbf{f})} \right\} &\quad (49) \\
&\overset{(a)}{=} \min_{S \subseteq V_S} \left\{ \min_{\mathbf{f} \in \mathcal{F} : \mathrm{gap}_S(\mathbf{f}) > 0} \left(\mathbf{1}_S^T \alpha\right) \frac{\Lambda^*(\mathbf{f})}{\mathrm{gap}_S(\mathbf{f})} \right\} . &\quad (50)
\end{aligned}$$

For completeness, define the minimum over the empty set as $+\infty$. Here (a) holds because: For a minimizer $\mathbf{f}^* \in \mathcal{F}$ of the outer problem of (49) and a minimizer $S^* \subseteq V_S$ of the inner problem of (49), $S^* \subseteq V_S$ is feasible for the inner problem of (50) while $\mathbf{f}^* \in \mathcal{F}$ is feasible for the outer problem of (50), hence (49) $\ge$ (50). Similarly we can show (49) $\le$ (50).

We claim that

$$(50) = \min_{J \in \mathcal{J}} \left\{ \min_{\mathbf{f} \in \mathcal{F} : \mathrm{gap}_{\partial(J)}(\mathbf{f}) > 0} \left(\mathbf{1}_{\partial(J)}^T \alpha\right) \frac{\Lambda^*(\mathbf{f})}{\mathrm{gap}_{\partial(J)}(\mathbf{f})} \right\} . \tag{51}$$

To see this, first note that for $S \subseteq V_S$ where $\{j' \in V_D : \partial(j') \subset S\}$ is empty, $\mathrm{gap}_S(\mathbf{f})$ is non-positive regardless of $\mathbf{f} \in \mathcal{F}$, hence such S can never be the minimizer. For other S, let $J \triangleq \{j' \in V_D : \partial(j') \subset S\}$, then $\partial(J) \subset S$. Note that

$$
\begin{aligned}
\mathrm{gap}_{\partial(J)}(\mathbf{f}) &= \sum_{j' \in J, k \in V_S} \mathbf{f}_{j'k} - \sum_{j' \in V_D, k \in \partial(J)} \mathbf{f}_{j'k} \\
&= \sum_{j' : \partial(j') \subset S, k \in V_S} \mathbf{f}_{j'k} - \sum_{j' \in V_D, k \in S} \mathbf{f}_{j'k} + \sum_{j' \in V_D, k \in S \setminus \partial(J)} \mathbf{f}_{j'k} \\
&= \mathrm{gap}_S(\mathbf{f}) + \sum_{j' \in V_D, k \in S \setminus \partial(J)} \mathbf{f}_{j'k} \\
&\geq \mathrm{gap}_S(\mathbf{f})\,.
\end{aligned}
$$

As a result, for $\mathbf{f}$ such that $\mathrm{gap}_S(\mathbf{f}) > 0$, we have

$$
\left(\mathbf{1}_S^T \alpha\right) \frac{\Lambda^*(\mathbf{f})}{\mathrm{gap}_S(\mathbf{f})} \geq \left(\mathbf{1}_{\partial(J)}^T \alpha\right) \frac{\Lambda^*(\mathbf{f})}{\mathrm{gap}_{\partial(J)}(\mathbf{f})}\,.
$$

Hence only those $S \subseteq V_S$ where $S = \partial(J)$ for $J \subseteq V_D$ can be the minimizer. If $J \notin \mathcal{J}$, then $\mathrm{gap}_{\partial(J)}(\mathbf{f}) \leq 0$ regardless of $\mathbf{f} \in \mathcal{F}$, so these sets are also ruled out. Therefore (51) holds.

Denote the optimal value of the inner minimization problem of (51) as $g(\phi, J) > 0$, then we have:

$$
\min_{\mathbf{f} \in \mathcal{F} : \mathrm{gap}_{\partial(J)}(\mathbf{f}) > 0} \Lambda^*(\mathbf{f}) - g(\phi, J) \left(\sum_{j' \in J, k \in V_S} f_{j'k} - \sum_{j' \in V_D, k \in \partial(J)} f_{j'k} \right) = 0\,. \tag{52}
$$

We can get rid of the constraint on $\mathbf{f}$ because for $\mathbf{f}$ where $\mathrm{gap}_{\partial(J)}(\mathbf{f}) \leq 0$, the argument of minimization in (52) is negative; and for $\mathbf{f} \notin \mathcal{F}$, $\Lambda^*(\mathbf{f}) = \infty$ by definition (6). Using Legendre transform, we have:

$$
\begin{aligned}
&\min_{\mathbf{f}} \Lambda^*(\mathbf{f}) - g(\phi, J) \left(\sum_{j' \in J, k \in V_S} f_{j'k} - \sum_{j' \in V_D, k \in \partial(J)} f_{j'k} \right) \\
&= \min_{\mathbf{f}} \Lambda^*(\mathbf{f}) - \mathbf{f}^T \left(g(\phi, J) \sum_{j' \in J, k \in V_S} \mathbf{e}_{j'k} - g(\phi, J) \sum_{j' \in V_D, k \in \partial(J)} \mathbf{e}_{j'k} \right) \\
&= -\Lambda \left(g(\phi, J) \sum_{j' \in J, k \in V_S} \mathbf{e}_{j'k} - g(\phi, J) \sum_{j' \in V_D, k \in \partial(J)} \mathbf{e}_{j'k} \right) \\
&\overset{(b)}{=} -\log \left(\sum_{j' \in V_D, k \in V_S} \phi_{j'k} e^{g(\phi,J)\mathbb{I}\{j' \in J\} - g(\phi,J)\mathbb{I}\{k \in \partial(J)\}} \right)\,.
\end{aligned}
$$

In (b) we use the fact that the dual function of $\Lambda^*(\mathbf{f})$ is $\Lambda(\mathbf{x}) = \log\left(\sum_{j' \in V_D, k \in V_S} \phi_{j'k} e^{x_{j'k}}\right)$ where $\mathbf{x} \in \mathbb{R}^{n \times n}$. Hence, using $\sum_{j' \in V_D, k \in V_S} \phi_{j'k} = 1$, Eq. (52) reduces to the nonlinear equation

$$
\left(\sum_{j' \notin J, k \in \partial(J)} \phi_{j'k} \right) e^{-g(\phi,J)} + \left(\sum_{j' \in J, k \notin \partial(J)} \phi_{j'k} \right) e^{g(\phi,J)} = \sum_{j' \notin J, k \in \partial(J)} \phi_{j'k} + \sum_{j' \in J, k \notin \partial(J)} \phi_{j'k}\,.
$$

Let $y \triangleq e^{g(\phi,J)}$, this becomes a quadratic equation:

$$\left(\sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}\right) y^2 - \left(\sum_{j'\notin J, k\in\partial(J)} \phi_{j'k} + \sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}\right) y + \left(\sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}\right) = 0\,.$$

Hence

$$y = \frac{\sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}}{\sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}} \text{ or } 1\,.$$

Since $g(\phi, J) > 0$, we have

$$g(\phi, J) = \log\left(\frac{\sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}}{\sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}}\right)\,.$$

Plugging into (51), we have:

$$\gamma_{\mathrm{AB}}(\alpha) = \min_{J\in\mathcal{J}} \left(\mathbf{1}^T_{\partial(J)}\alpha\right) \log\left(\frac{\sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}}{\sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}}\right)\,.$$

Most likely demand sample path leading to demand drop. Denote

$$\mathbf{c} \triangleq g(\phi, J)\left(\sum_{j'\in J, k\in V_S} \mathbf{e}_{j'k} - \sum_{j'\in V_D, k\in\partial(J)} \mathbf{e}_{j'k}\right)\,,$$

denote $\mathbf{f}_J$ as the minimizer of the LHS of (51):

$$\mathbf{f}_J = \mathrm{argmin}_{\mathbf{f}\in\mathcal{F}} \sum_{j'\in V_D} \sum_{k\in V_S} \left(\mathbf{f}_{j'k} \log\frac{\mathbf{f}_{j'k}}{\phi_{j'k}} - c_{j'k}\mathbf{f}_{j'k}\right)\,.$$

First order condition implies: $(\mathbf{f}_J)_{j'k} = \phi_{j'k}\frac{e^{c_{j'k}}}{\sum_{j',k}\phi_{j'k}e^{c_{j'k}}}$. Recall the definition of λ_J, μ_J in (11), we have

$$\begin{aligned}
\sum_{j',k}\phi_{j'k}e^{c_{j'k}} &= \sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}\frac{\lambda_J}{\mu_J} + \sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}\frac{\mu_J}{\lambda_J} + \left(1 - \sum_{j'\in J, k\notin\partial(J)} \phi_{j'k} - \sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}\right) \\
&= \mu_J\frac{\lambda_J}{\mu_J} + \lambda_J\frac{\mu_J}{\lambda_J} + (1 - \lambda_J - \mu_J) \\
&= 1\,.
\end{aligned}$$

Hence

$$(\mathbf{f}_J)_{j'k} = \begin{cases} \phi_{j'k}(\lambda_J/\mu_J), & \text{for } j'\in J, k\notin\partial(J) \\ \phi_{j'k}(\mu_J/\lambda_J), & \text{for } j'\notin J, k\in\partial(J) \\ \phi_{j'k}, & \text{otherwise} \end{cases}\,.$$

Let $J^* = \mathrm{argmin}_{J\in\mathcal{J}} B_J \log(\lambda_J/\mu_J)$, then demand sample path with constant derivative $\mathbf{f}_{J^*}$ is the most likely sample path leading to demand drop. $\square$

E.4 Performance of Vanilla MaxWeight: Proof of Proposition 4

Proof of Proposition 4. We have

$$\text{LHS} = \min_{J\in\mathcal{J}} \frac{|\partial(J)|}{n} \log\left(\frac{\sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}}{\sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}}\right), \tag{53}$$

$$\gamma^* = \max_{\alpha\in\Omega} \min_{J\in\mathcal{J}} (\mathbf{1}^{\mathrm{T}}_{\partial(J)}\alpha) \log\left(\frac{\sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}}{\sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}}\right). \tag{54}$$

Let $J^* \subsetneq V_D$ be one of the minimizer of (53), α^* is the maximizer of (54). Hence

$$\begin{aligned}
\gamma^* &= \min_{J\in\mathcal{J}} (\mathbf{1}^{\mathrm{T}}_{\partial(J)}\alpha^*) \log\left(\frac{\sum_{j'\notin J, k\in\partial(J)} \phi_{j'k}}{\sum_{j'\in J, k\notin\partial(J)} \phi_{j'k}}\right) \\
&\le (\mathbf{1}^{\mathrm{T}}_{\partial(J^*)}\alpha^*) \log\left(\frac{\sum_{j'\notin J^*, i\in\partial(J^*)} \phi_{j'k}}{\sum_{j'\in J^*, i\notin\partial(J^*)} \phi_{j'k}}\right) \\
&\le |\partial(J^*)| \log\left(\frac{\sum_{j'\notin J^*, i\in\partial(J^*)} \phi_{j'k}}{\sum_{j'\in J^*, i\notin\partial(J^*)} \phi_{j'k}}\right) \\
&= n\gamma\left(\frac{1}{n}\mathbf{1}\right).
\end{aligned}$$

□

F Simulation Settings

Model Primitives.

- *Demand arrival process (ϕ).* Using the estimation in Buchholz (2015), which is based on Manhattan's taxi trip data during August and September in 2012, we obtain the (average) demand arrival rates for each origin-destination pair during the day (7 a.m. to 4 p.m.) denoted by $\tilde{\phi}_{ij}$ ($i, j = 1, \cdots, 30$). However, we find that $\tilde{\phi}_{ij}$ violates CRP (there are a lot more rides *to* Midtown than *from* Midtown). We consider the following "symmetrization" of $\tilde{\phi} \triangleq (\tilde{\phi}_{ij})_{30\times 30}$ to ensure that CRP holds (ride-hailing platforms may use spatially varying prices and repositioning to obtain CRP, see Section 1):

$$\phi(\eta) \triangleq \eta\tilde{\phi} + (1-\eta)\frac{1}{2}(\tilde{\phi} + \tilde{\phi}^{\mathrm{T}}), \qquad \eta \in (0, 1). \tag{55}$$

  Figure 10 shows how the Hall's gap of $\phi(\eta)$ varies with η. We pick $\eta = 0.21$ such that CRP is "almost violated"[13]. The subset of locations with smallest Hall's gap is then the Upper West Side (locations 19, 23, 24, 27, 28 in Figure 5).
- *Pickup/service times ($D/\tilde{D}$).* We extract the pairwise travel time between region centroids (marked by the dots in Figure 5) using Google Maps, denoted by D_{ij}'s ($i, j = 1, \cdots, 30$). We use D_{ij} as service time for customers traveling from i to j. For each customer at i who is picked up by a supply from k we add a pickup time [14] of $\tilde{D}_{ki} = \max\{\frac{3}{2}D_{ki}, 3 \text{ minutes}\}$.

[13] We also ran simulations for $\eta = 0.15$ such that Hall's gap is large. There is no significant difference in the policies' relative performances, so we didn't include it here.

[14] We use the inflated D_{ij}'s as pickup times to account for delays in finding or waiting for the customer.

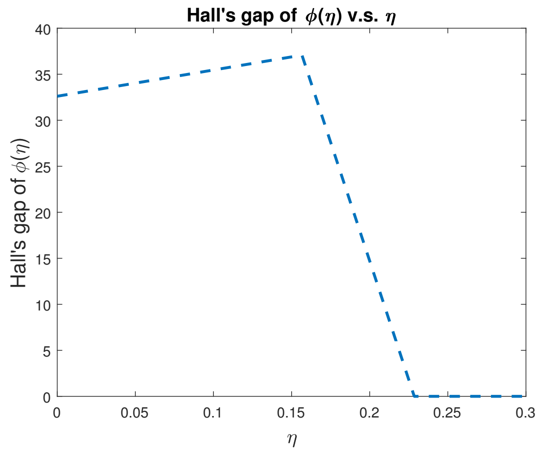

Figure 10: Hall's gap of symmetrized matrix $\phi(\eta)$ (see Eq. (55)) versus parameter η, based on the demand arrival rates $\tilde{\phi}$ computed from the Manhattan taxi data. Our simulations use $\eta = 0.21$, which corresponds to a small but non-zero Hall's gap (< 10).

Benchmark policy: fluid-based policy. We consider the fluid-based randomized policy (Banerjee et al. 2016, Ozkan and Ward 2016) as a benchmark. Let $\mathcal{X}$ be the solution set of the feasibility problem

$$\sum_{j\in\partial(i)} x_{ij} = \lambda_i \quad \forall i, \qquad \sum_{i\in\partial(j)} x_{ij} = \mu_j \quad \forall j.$$

Since CRP holds, $\mathcal{X} \neq \emptyset$. Let $\mathbf{x}^* \triangleq \operatorname{argmin}_{\mathbf{x}\in\mathcal{X}} \sum_{(i,j)\in E} \tilde{D}_{ij} x_{ij}$. When demand arrives at location j, the randomized fluid-based policy dispatches from location $i \in \partial(j)$ with probability x^*_{ij}/μ_j. Then x^*_{ij} is the rate of dispatching cars from i to serve demand at j. From Banerjee et al. (2016), we know that $\mathbf{x}^*$ leads to a zero demand-drop as $K \to \infty$ with and without pickup times (assuming demand remains constant). Moreover, with pickup times, Little's Law gives that the fluid-based policy minimizes the expected number of cars on-route to pick up customers.

Benchmark fleet-size. In the *Service time* setting, a fraction of cars are in transit under the stationary distribution; in the *Service+Pickup time* setting, there is an additional fraction of cars on-route to pick up customers. A simple workload conservation argument (using Little's Law) gives the benchmark fleet-sizes as follows.

- *Service time.* Assuming no demand is dropped, the mean number of cars in transit is: $K_{\text{fl}} = \sum_{i,j} \phi_{ij} D_{ij}$. In our setting, we have $K_{\text{in-transit}} \approx 7{,}061$. Since CRP holds and demand-drop probability goes to 0 under both fluid-based policy and SMW policies, $K_{\text{in-transit}}$ is a reasonable benchmark fleet-size K_{fl}. We will vary the number of cars in the system denoted by $K_{\text{tot}} = K_{\text{fl}} + K_{\text{slack}}$ and compare the performance of different policies. Here K_{slack} is the number of free cars in the system when no demand is dropped.

- *Service+Pickup time.* Applying Little's Law, if no demand is dropped, the mean number of cars picking up customers is at least $K_{\text{pickup}} = \min_{\mathbf{x}\in\mathcal{X}} \tilde{D}_{ij} x_{ij}$. In our case, we have $K_{\text{pickup}} \approx 2{,}941$. Hence, the benchmark fleet size is $K_{\text{fl}} = K_{\text{in-transit}} + K_{\text{pickup}} = 10{,}002$.

Note that this number is close to the real-world fleet size: there were approximately 11,500 active medallions when Buchholz (2015) was written.