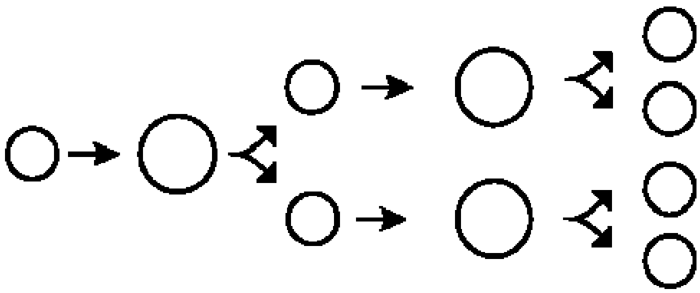
Exponential growth

Say we start with one cell, put it in minimal medium, where it and its daughter cells will grow and divide once every hour:



In minimal medium , E. coli divides typically in 60 min., or 1 generation= 60 min.

We can calculate how long it will take to get a billion cells from just one:

Let g = number of generations. 2 gens. --> 4 cells, 3 gens. --> 8 cells, or N (no. of cells) $=1 \times 2^g$ (starting with one cell).
If we started with 100 cells, after 1 gen. we would have 200 and after 2 gens. 400, 3 gens., 800 etc.

More generally, starting with N_o cells: $N = N_o \times 2^g$

Since we want to know how much <u>time</u> it will take: express generations in terms of time.
If we let t_D = the generation time, or <u>d</u>oubling time, then the number of generations that have passed during the time interval t is just t/t_D: So $g = t/t_D$.
So now $N = N_o \times 2^{t/tD}$. One can thus see that growth is exponential with respect to time.

Now we could solve this equation for t, since we know we want N to be 1 billion, N_o is 1, and t_D is 1 hr. Taking the logarithm base 2 of both sides:

$\log_2 (N/N_o) = t/t_D$, or $t = t_D\log_2(N/N_o) = 1 \times \log_2 (1{,}000{,}000{,}000/1) = \log_2(10^9)$

But suppose your calculator doesn't do log base 2. No problem, convert to log base 10 ("log") or natural log base e ("ln").

$\log_2X = \log X/\log 2 = \log X/0.3$ and $\log_2X = \ln_eX/\ln_e2 = \ln X/0.69$ (Also: $2^x = 10^{x\log 2}$ and $2^x = e^{x\ln 2}$)

So $\log_2(N/N_o) = \log(N/N_o)/\log 2 = t/t_D$ or $\log(N/N_o) = (\log 2/t_D)t = Kt$, where $K=\log 2/t_D$ or $K=0.3/t_D$.

Or back to the exponential form: $N/N_o = 10^{Kt}$ or: $N = N_o 10^{Kt}$

Or, since most scientific calculators have natural log functions:

$N = N_oe^{Kt}$, where $K = \ln 2/t_D = 0.69/t_D$, another common form of the exponential growth equation.

We could also have approached this question of rates of change of N with time more naturally using calculus (Note: familiarity with calculus is not necessary for this course.) If you have a million cells, then after one generation time you'll have gained 1 million. If you had 100, you would've gained 100. In general, the rate of increase of N with time is just proportional to the number of cells you have at any moment in time, or: $dN/dt = KN$

Separating variables: $dN/N = Kdt$.
Integrating between time zero when $N = N_o$ and time t, when $N = N$:

$\ln N - \ln N_o = Kt - 0$, or $\ln(N/N_o) = Kt$, or $N = N_o e^{Kt}$

We can calculate the constant K by considering the time interval over which N_o has doubled. This time is the doubling time, t_D. For that condition:
$N/N_o = 2 = e^{Kt_D}$. Taking the natural logarithm of both sides: $\ln 2 = Kt_D$, or $K = \ln 2/t_D$, exactly as above.

In summary:

	Base 2	Base 10	Base e
Exponential form	$N = N_o 2^{K_2 t}$	$N = N_o 10^{K_{10} t}$	$N = N_o e^{K_e t}$
Logarithmic form	$\log_2(N/N_o) = K_2 t$	$\log(N/N_o) = K_{10} t$	$\ln(N/N_o) = K_e t$
Definition of constant	$K_2 = 1/t_D$	$K_{10} = \log_{10} 2/t_D = 0.3/t_D$	$K_e = \ln 2/t_D = 0.69/t_D$

All this looks worse than it is. Exponential growth using a base of 2 is intuitively obvious. And once you see the derivation, the exponential growth equation using log or ln can be simply applied to problems using a calculator. You just have to keep track of what you know and what you are after. Graphically, the depiction of exponential growth looks like graph A. Or, with the ordinate (Y-axis) plotted on a logarithmic scale, a semi-log plot, B. In reality, there's a lag before cells get going, and there's a limit (thankfully) to cell density, as nutrients become exhausted and/or toxic excretions accumulate. The final plateau is called stationary phase (graph C).