Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 12, 2007
9:00 AM – 11:00 AM

General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5(General Physics) Question 7, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Clouds of *CN* molecules are observed in interstellar space. Each molecule can be modeled as a rigid rod, with a separation between the carbon and nitrogen nuclei of 1.17Å.

(a) Find the approximate energy spectrum associated with rotational excitations of *CN*. You can assume the most common isotopes are ^{12}C and ^{14}N.

(b) At a temperature of 2.7 K, what is the fraction of the *CN* molecules that should be in their ground state? Assume that the molecules do not dissociate.

(c) How would your answers in parts (a) and (b) change for an N_2 molecule? The ^{14}N nucleus has spin 1, and the spacing between nuclei in an N_2 molecule is 1.1Å.

Useful facts: $\hbar c \approx 2 \times 10^3$ eV·Å and $k_B \approx 8.6 \times 10^{-5}$ eV/K.

2. Consider a gas of N nonrelativistic fermions with spin ½ and mass m originally confined in a volume V_0 and kept at zero temperature.

(a) Express the kinetic energy of the gas in terms of N and V_0.

(b) What is the pressure of the gas? Assume that the gas is ideal.

(c) Now the gas is allowed to expand to the volume $V_1 >> V_0$ without any energy exchange with the outside world. Calculate the temperature of the gas after it reaches an equilibrium due to weak interactions between the fermions.

3. (a) Estimate the temperature T at the center of the Sun, which has a mass $M = 2 \times 10^{33}$ g and a radius $R = 7 \times 10^{10}$ cm. Make a crude estimate, neglecting all numerical factors like 4π.

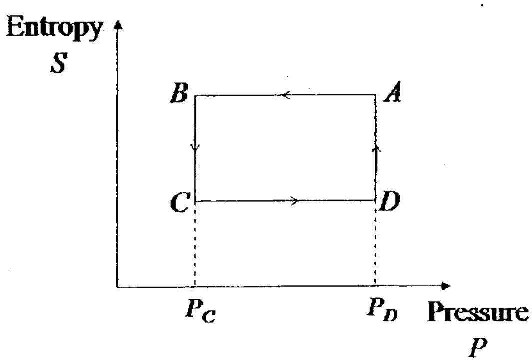
(b) Explain why the estimate changes for stars with

$$M >> M_{\bullet} = \frac{\mathsf{R}_g^2}{a^{1/2} G^{3/2}} \sim 10^{34} \text{ g}$$

and estimate T for this case.

$G = 6.7 \times 10^{-8}$ cm^3 s^{-2} g^{-1} is the gravitational constant,
$\mathsf{R}_g = 8.3 \times 10^7$ erg K^{-1} g^{-1} is the specific gas constant for hydrogen,
$a = 7.6 \times 10^{-15}$ erg cm^{-3} K^{-4} is the radiation constant.

4.



The reversible Brayton engine cycle $A \to B \to C \to D \to A$ is shown in the entropy-pressure (*S*-*P*) diagram. During step $D \to A$, the gas is heated at constant pressure with the absorption of an amount of heat Q_{DA}. During step $B \to C$, the gas is cooled at constant pressure with the expulsion of an amount of heat Q_{BC}. Let W_{ABCD} denote the net work done by the gas during a complete cycle.

The efficiency of the engine is defined to be the ratio W_{ABCD} / Q_{DA} (because the expelled heat Q_{BC} is considered to be wasted). Consider an ideal gas as the working substance, so that the heat capacities C_P and C_V are temperature independent.

(a) Show this cycle in a pressure-volume (*P*-*V*) diagram.

(b) Calculate the efficiency of the engine in terms of C_P, C_V, and the pressure ratio P_C / P_D.

5. (a) How much ice at 0°C do you need to make iced coffee at $T_f = 5°C$ from 500 g of coffee at $T_i = 80°C$? The heat capacity of coffee is $c = 1$ cal/(g °C), the latent heat of fusion for ice is $L_f = 80$cal/g. You can assume that the coffee mug is thermally isolated.

(b) If you have no ice, but only water at 0°C, how much of it do you need to cool the coffee to T_f?

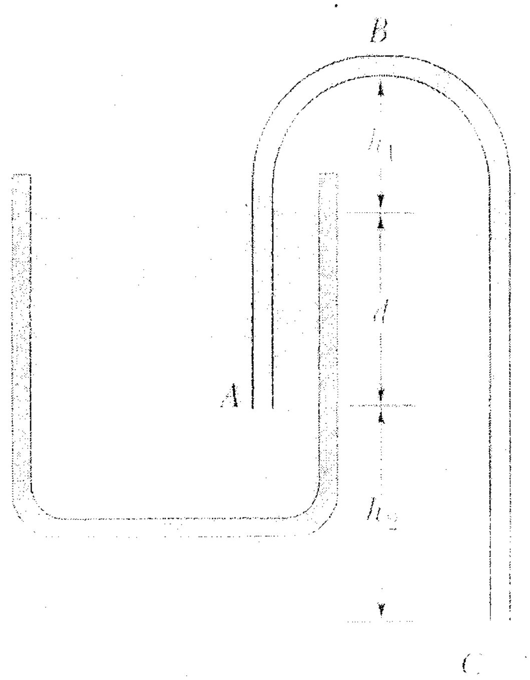
6. A siphon is a device to remove liquid from a container which cannot be tipped. The tube must be completely filled with water to start the siphon, but once this is achieved, the liquid will flow through the tube. Consider the siphon shown below and assume that the container is large compared to the tube.

(a) Show that the water emerges at C with velocity

$$v_c \approx \sqrt{2g(d+h_2)}$$

(b) Calculate the pressure at point B for a siphon with h_1 = 25 cm, d = 12 cm, and h_2 =40 cm. The density of water is 1000 kg/m^3, and you can neglect viscosity. The atmospheric pressure is 1.0×10^5 Pa.

(c) What is the maximum height h_1 that this siphon can lift water ?



on 11/16/06 Sec 5 #1

$$\mu = \frac{m_c m_N}{m_c + m_N} \approx \frac{1}{2} \cdot 13\,GeV/c^2$$

KABAT question soluti

$$H = \frac{L^2}{2\mu r^2}$$

Quantum

$$E_L = \frac{\hbar^2 L(L+1)}{2\mu r^2} \qquad L = 0, 1, 2, \ldots$$

function $Z = e^{-\beta E_0} + 3e^{-\beta E_1}$

rigid rot. state) $= \dfrac{e^{-\beta E_0}}{e^{-\beta E_0} + 3e^{-\beta E_1}} = \dfrac{1}{1 + 3e^{\beta(E_0 - E_1)}}$

spectrum : $kT = 8.6\times10^{-5}\,eV/K \cdot 2.7K = 2.3\times10^{-4}\,eV$

$$\frac{\hbar^2 \cdot 2}{2\mu r^2} = \frac{\hbar^2 c^2}{\mu c^2 r^2} = \frac{(2\times10^3\,eV\,\mathring{A})^2}{6.5\times10^9\,eV\cdot(1.2\mathring{A})^2} = 4\times10^{-4}\,eV$$

prob (ground) $= \dfrac{1}{1 + 3e^{-4\times10^{-4}/2.3\times10^{-4}}} = 68\%$

the numbers we have the same $E_0 - E_1$ roughly, and

$-E_0 + E_1$ = spin 0 or 2 ← six symmetric spin states, can be in $L=0$

probabili

spin 1 ← three antisymmetric spin states can be in $L=1$

For N_2

total lity $= \dfrac{6e^{-\beta E_0}}{6e^{-\beta E_0} + 3\times3e^{-\beta E_1}} = \dfrac{1}{1 + \frac{3}{2}e^{-4\times10^{-4}/2.3\times10^{-4}}}$

$= 79\%$

total

⇒ probab

ALTSCHULER Sec 5 # 2

Solution of the Problem on Statistical Physics

Consider a gas of N nonrelativistic fermions with spin 1/2 and mass m originally confined in a volume V_0 and kept at zero temperature.

1. Express the kinetic energy of the gas in terms of N and V_0. (4 points)

At zero temperature the fermions occupy all the states inside the Fermi sphere in the momentum space. To evaluate the radius of the Fermi sphere p_F, one has to recall that the density of states in the phase space for a given direction of the spin is $(2\pi\hbar)^{-3}$, i.e. the volume of the Fermi sphere should be equal to the density of the fermions with given spin times $(2\pi\hbar)^3$

$$\frac{4}{3}\left(\frac{p_F}{2\pi\hbar}\right)^3 = \frac{N}{2V_0} \qquad p_F = (2\pi\hbar)\left(\frac{3N}{8V_0}\right)^{1/3}$$

In order to evaluate the total kinetic energy one has to calculate the integral over the volume inside the Fermi sphere

$$E_K = 2V_0 \int_{|\vec{p}|\le p_F} \frac{d^3p}{(2\pi\hbar)^3}\frac{p^2}{2m}$$

(factor 2 reflects two possible directions of the spin). As a result

$$E_K = \frac{V_0}{\pi^2\hbar^3}\frac{p_F^5}{10m} = \frac{3^{5/3}}{160\pi\hbar^2 m}\left[\frac{N^5}{V_0^2}\right]^{1/3}$$

2. What is the pressure of the gas? You can assume here that the gas is ideal. (3 points)

Assuming that the gas is ideal we can neglect the energy of the interaction between the fermions and equate the total energy to the kinetic energy. The pressure of an ideal gas at zero temperature can be determined as a derivative of the kinetic energy over the volume

$$P_0 = -\frac{\partial E_K}{\partial V_0} = \frac{3^{2/3}}{80\pi\hbar^2 m}\left[\frac{N}{V_0}\right]^{5/3}$$

3. Now the gas is allowed to expand to the volume $V_1 >> V_0$ without any energy exchange with the outside world. Calculate the temperature of the gas after it will reach the equilibrium due to weak interactions between the fermions. (5 points)

The expansion of the gas without energy exchange with the outside world conserves the total energy, and the energy per particle also remains the same.

Provided that $V_1 >> V_0$ the gas in the final state is degenerated and its temperature times Boltzmann constant is of the order of the energy per particle:

$$k_B T \sim \frac{E_K}{N} = \frac{3^{5/3}}{160\pi\hbar^2 mN}\left[\frac{N^5}{V_0^2}\right]^{1/3} = \frac{3^{5/3}}{160\pi\hbar^2 m}\left[\frac{N}{V_0}\right]^{2/3}$$

This is much bigger than the Fermi energy on N particles in the volume V_I(this Fermi energy is $\propto V_1^{-2/3}$), i.e., the gas in the final state is indeed nondegenerated. In order to determine the prefactor one has to evaluate the distribution function in the final state, which is characterized by the temperature T and the chemical potential μ:

$$n_F(p) = \left[\exp\left(\frac{\frac{p^2}{2m} - \mu}{k_B T}\right) + 1\right]^{-1}$$

In general the parameters of the distribution in the final state, T and μ., should be determined from the given number of particles and given kinetic energy:

$$\frac{N}{2V_1} = \int n_F(p)\frac{d^3p}{(2\pi\hbar)^3} \qquad E_K = V_1\int\frac{p^2}{2m}n_F(p)\frac{d^3p}{(2\pi\hbar)^3}$$

In the limit $V_1 >> V_0$ we can avoid solving these two equations: we expect Boltzmann distribution in the final state:

$$n(p) = \frac{N}{2V_1}\left(\frac{2\pi\hbar^2}{mk_B T}\right)^{3/2} \exp\left(\frac{-p^2}{2mk_B T}\right)$$

(To check the numerical factor in this distribution one has to recall that $\int_0^\infty dx e^{-x}\sqrt{x} = \frac{1}{2}\sqrt{\pi}$). Now we can use this distribution to connect the temperature with the energy:

$$E_K = V_1\frac{N}{2V_1}\left(\frac{2\pi\hbar^2}{mk_B T}\right)^{3/2}\int\frac{p^2}{2m}n(p)\frac{d^3p}{(2\pi\hbar)^3} = \frac{N}{\sqrt{\pi}}k_B T\int_0^\infty dx e^{-x}x^{3/2} = \frac{3}{4}k_B TN$$

(we introduced dimensionless variable $x = p^2/(2mk_B T)$ and took into account that $\int_0^\infty dx e^{-x}x^{3/2} = \frac{3}{4}\sqrt{\pi}$). Substituting the expression for the kinetic energy

$$E_K = \frac{V_0}{\pi^2\hbar^3}\frac{p_F^5}{10m} = \frac{3^{5/3}}{160\pi\hbar^2 m}\left[\frac{N^5}{V_0^2}\right]^{1/3}$$

we obtain the temperature

$$k_B T = \frac{3^{2/3}}{40\pi\hbar^2 m}\left[\frac{N}{V_0}\right]^{2/3}$$

4. What is the pressure of the gas in the final state? (3 points)

One can use the ideal gas law $P_1 V_1 = N k_B T$ and obtain

$$P_1 = \frac{N}{V_1} k_B T = \frac{3^{2/3}}{40 \pi \hbar^2 m} \left[\frac{N^5}{V_0^{\,2} V_1^{\,3}} \right]^{1/3}$$

SEC 5 # 3

Problem 3.

(a) Estimate temperature T at the center of the Sun, which has mass $M = 2 \times 10^{33}$ g and radius $R = 7 \times 10^{10}$ cm. Make a crude estimate, neglecting all numerical factors like 4π.

(b) Explain why the estimate changes for stars with

$$M \gg M_* = \frac{\mathcal{R}_g^2}{a^{1/2} G^{3/2}} \sim 10^{34} \text{ g},$$

and estimate T for this case.

$G = 6.7 \times 10^{-8}$ cm^3 s^{-2} g^{-1} is the gravitational constant,
$\mathcal{R}_g = 8.3 \times 10^7$ erg K^{-1} g^{-1} is the gas constant,
$a = 7.6 \times 10^{-15}$ erg cm^{-3} K^{-4} is the radiation constant.

Solution:

(a) The Sun is supported by pressure P against its gravity,

$$R^2 P \sim \frac{GM^2}{R^2}, \qquad P \sim \rho \mathcal{R}_g T.$$

Density ρ may be roughly estimated as $\rho \sim M/R^3$, so

$$\frac{M}{R} \mathcal{R}_g T \sim \frac{GM^2}{R^2}, \qquad T \sim \frac{GM\mathcal{R}_g}{R}. \tag{1}$$

This gives $T \sim 10^7$ K for the solar mass and radius.

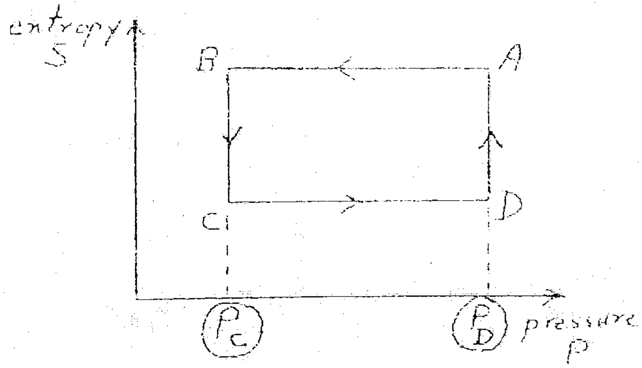
(b) Besides the gas, the blackbody radiation contributes to the pressure, $P_{rad} = \frac{1}{3} a T^4$. If radiation dominates the total pressure, the hydrostatic balance gives

$$R^2 a T^4 \sim \frac{GM^2}{R^2}, \qquad T \sim \left(\frac{G}{a}\right)^{1/4} \frac{M^{1/2}}{R}. \tag{2}$$

Comparing equations (1) and (2) one finds the characteristic mass M_* above which P_{rad} dominates and T is given by equation (2) instead of (1).

① Thermodynamics (general exam)

7. (30 points)



The reversible Brayton engine cycle A → B → C → D → A is shown in the entropy-pressure (S-P) diagram. During step D → A, the gas is heated at constant pressure with the absorption of an amount of heat Q_{DA}. During step B → C, the gas is cooled at constant pressure with the expulsion of an amount of heat Q_{BC}.

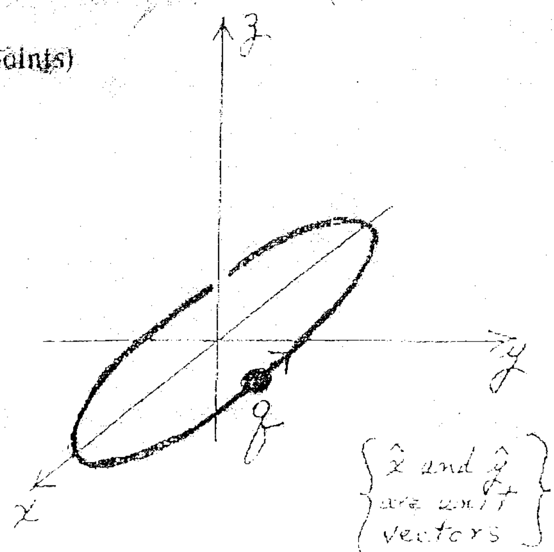
Let W_{ABCD} denote the net work done by the gas during a complete cycle. The efficiency of the engine is defined to be the ratio W_{ABCD}/Q_{DA} (because the expelled heat Q_{BC} is considered to be wasted). Consider an ideal gas as the working substance, so that the heat capacities C_P and C_V are temperature independent.

(a) Show this cycle in a pressure-volume (P-V) diagram.

(b) Calculate the efficiency of the engine in terms of C_P, C_V, and the pressure ratio P_C/P_D.

② Electromagnetism (general exam)

8. (30 points)



A particle of charge q moves in the x-y plane along an elliptical orbit, as shown in the diagram. The particle's position vector $\vec{s}$ depends on time t:

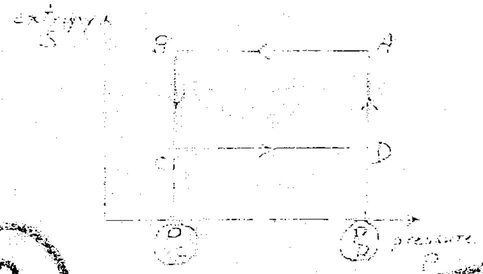
$$\vec{s}(t) = \hat{x} R_1 \cos(\omega t) + \hat{y} R_2 \sin(\omega t)$$

where ω, R_1, and R_2 are given positive constants. Assume $R_1 > R_2$. Also, assume that the particle is in non-relativistic motion (v << c).

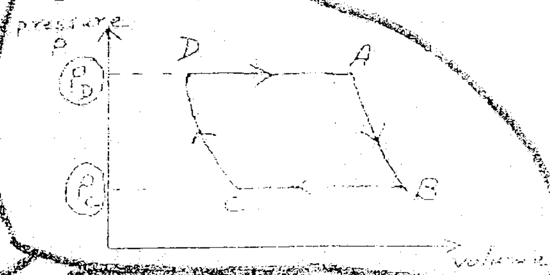
(a) Find the time-averaged power radiated (emitted) per unit solid angle ($dP/d\Omega$)$_{avg}$ in the direction specified by the polar and azimuthal angles θ and φ.

(b) Find the polar angle θ of the field point for which the detected radiation is circularly polarized. Consider only field points in the first quadrant of the x-z plane (φ = 0 and 0 < θ < π/2). Briefly explain your reasoning.

30 total points



(a)



10 pts

$A \to B$ and $C \to D$ have constant entropy (no heat absorbed or expelled). $\therefore$ $A \to B$ and $C \to D$ are <u>adiabatic</u> processes with $pV^{\gamma} = \text{constant}$, where $\gamma \equiv c_p/c_V$.

2 pts.

(b) Efficiency $\eta \equiv \dfrac{W_{ABCD}}{Q_{DA}}$.

2 pts

$$\underbrace{\delta Q}_{\substack{\text{heat added to}\\ \text{system}}} = \underbrace{\delta W}_{\substack{\text{work}\\ \text{done by}\\ \text{system}}} + \underbrace{dU}_{\substack{\text{increase}\\ \text{in system's}\\ \text{internal energy}}} \implies Q_{ABCD} = W_{ABCD} \text{ since } \Delta U = 0 \text{ for a complete cycle.}$$

$$Q_{ABCD} = \underbrace{Q_{AB}}_{0} + Q_{BC} + \underbrace{Q_{CD}}_{0} + Q_{DA} = Q_{BC} + Q_{DA}.$$

$$\therefore \eta = \frac{Q_{BC} + Q_{DA}}{Q_{DA}} \implies \boxed{\eta = 1 + \frac{Q_{BC}}{Q_{DA}}}$$

Q_{BC} (negative)
Q_{DA} (positive)

<u>Note</u>: $\delta Q = p\,dV + dU \implies (\delta Q)_V = (dU)_V$

$$C_V \equiv \left(\frac{\delta Q}{\delta T}\right)_V = \left(\frac{dU}{dT}\right)_V \implies \boxed{U = C_V T}$$

2 pts

because C_V is independent of T and because $U = 0$ at $T = 0$.

2 pts

D → A process

$$Q_{DA} = \underbrace{W_{DA}}_{\underbrace{\int_{V_D}^{V_A} p\,dV}_{p_D(V_A - V_D)}} + \underbrace{\Delta U_{DA}}_{\underbrace{C_V(T_A - T_D)}_{\frac{C_V}{Nk} p_D (V_A - V_D)}}$$

Ideal Gas $pV = NkT$

$\therefore T_A = \dfrac{p_A V_A}{Nk} = \dfrac{p_D V_A}{Nk}$

& $T_D = \dfrac{p_D V_D}{Nk}$

2 pts

Problem 1 continued

Problem # continued

$$\therefore \boxed{Q_{DA} = \left(1+\frac{C_V}{Nk}\right) P_D (V_A - V_D)}.$$

B → C process

$$Q_{BC} = \underbrace{W_{BC}}_{\underbrace{\int_{V_B}^{V_C} p\,dV}_{-P_C(V_B-V_C)}} + \underbrace{\Delta U_{BC}}_{\underbrace{C_V(T_C-T_B)}_{-\frac{C_V}{Nk}P_C(V_B-V_C)}}$$

Ideal Gas $pV = NkT$

$\therefore T_C = \frac{P_C V_C}{Nk}$

& $T_B = \frac{P_B V_B}{Nk} = \frac{P_C V_B}{Nk}$

$$\therefore \boxed{Q_{BC} = -\left(1+\frac{C_V}{Nk}\right) P_C (V_B - V_C)}.$$

Method 8 pts

Now, $\eta = 1 + \frac{Q_{BC}}{Q_{DA}} \Longrightarrow \eta = 1 - \frac{P_C(V_B - V_C)}{P_D(V_A - V_D)}$

A → B is adiabatic $\Longrightarrow \underbrace{P_A}_{P_D} V_A^{\gamma} = \underbrace{P_B}_{P_C} V_B^{\gamma} \Longrightarrow P_D^{\frac{1}{\gamma}} V_A = P_C^{\frac{1}{\gamma}} V_B$

C → D is adiabatic $\Longrightarrow P_C V_C^{\gamma} = P_D V_D^{\gamma} \Longrightarrow P_C^{\frac{1}{\gamma}} V_C = P_D^{\frac{1}{\gamma}} V_D$

Thus, $P_C^{\frac{1}{\gamma}}(V_B - V_C) = P_D^{\frac{1}{\gamma}}(V_A - V_D) \Longrightarrow \frac{(V_B - V_C)}{(V_A - V_D)} = \left(\frac{P_D}{P_C}\right)^{\frac{1}{\gamma}}$

$$\therefore \eta = 1 - \frac{P_C}{P_D}\left(\frac{P_D}{P_C}\right)^{\frac{1}{\gamma}} = 1 - \left(\frac{P_C}{P_D}\right)^{1-\frac{1}{\gamma}}$$

$\gamma \equiv \frac{C_p}{C_V} \Rightarrow \left(1 - \frac{1}{\gamma}\right) = \left(1 - \frac{C_V}{C_p}\right) = \left(\frac{C_p - C_V}{C_p}\right)$ $[C_p > C_V]$

Answer 2 pts

$$\therefore \boxed{\eta = 1 - \left(\frac{P_C}{P_D}\right)^{\left(\frac{C_p - C_V}{C_p}\right)}}.$$

Sec 5 #5

Westerhoff
solution
1/5/07

Section 5, General I

Question #5 (Ice Coffee)

(1) Since the coffee mug is thermally isolated, the net heat flow to the outside is zero and all heat flow is within the system:

$$\Delta Q_1 + \Delta Q_2 + \Delta Q_3 = 0$$ (5)

melt ice

raise temperature of melted ice to T_f

Lower temperature of water to T_f

$\Delta Q_1 = m_1 L_f$ $\quad$ m_1 mass of ice

$\Delta Q_2 = c\, m_1 (T_f - 0°C)$ $\quad$ m_2 mass of water

$\Delta Q_3 = c\, m_2 (T_f - T_i)$ (10)

$$\Rightarrow \quad m_1 L_f + m_1 c\, T_f + m_2 c\,(T_f - T_i) = 0$$

$$\Leftrightarrow \quad m_1 = \frac{-m_2 c\,(T_f - T_i)}{L_f + c\,T_f}$$

$$= \frac{500\,g \; 1\,\frac{cal}{g\,°C} \; 75\,°C}{80\,\frac{cal}{g} + 1\,\frac{cal}{g\,°C}\;5\,°C}$$

$$= \underline{\underline{441\,g}}$$ (8)

(2) now we have

$$m_1' = \frac{-m_2 c \,(T_f - T_i)}{c\, T_f} = -m_2 \frac{T_f - T_i}{T_f} \qquad (7)$$

$$= -m_2 \left(1 - \frac{T_i}{T_f}\right) = \underline{\underline{7500}}\ g$$

sec 5 #6

Westerhoff - Solution
1/5/07

Section 5, General I

Question #6 (Siphon)

(1) Consider a point D on the surface of the liquid and apply Bernoulli's equation to D and C:

$$P_D + \frac{1}{2}\rho v_D^2 + \rho g h_D = P_c + \frac{1}{2}\rho v_c^2 + \rho g h_c \qquad (5)$$

$$\Rightarrow \quad v_c = \sqrt{\frac{2(P_D - P_c)}{\rho} + 2g(h_D - h_c) + v_D^2} \qquad (5)$$

now use $P_D \simeq P_c \simeq P_{air}$ and $v_D \ll v_c$ for a large container (5)

and $h_D - h_c = d + h_2 \quad \Rightarrow \quad v_c \simeq \sqrt{2g(d+h_2)}$

(2) Now consider points B and C and apply Bernoulli:

$$P_B + \frac{1}{2}\rho v_B^2 + \rho g h_B = P_c + \frac{1}{2}\rho v_c^2 + \rho g h_c \qquad (5)$$

and use $v_B = v_c$ (continuity equation) and $P_c = P_{air}$

$$P_B = P_{air} + \rho g (h_c - h_B)$$

$$= P_{air} - \rho g (d + h_1 + h_2)$$

$$= 1.0 \cdot 10^5\ \text{Pa} - 1.0 \cdot 10^3\ \frac{kg}{m^3} \cdot 9.8\ \frac{m}{s^2} \cdot 0.77\ m$$

$$= \underline{\underline{9.2 \cdot 10^4\ \text{Pa}}} \qquad (5)$$

(3) condition is $P_B \geq 0$, So

$$P_{air} - \rho g (d + h_1 + h_2) \geq 0$$

$$\Rightarrow \quad h_1 \leq \frac{P_{air}}{\rho g} - d - h_2 \simeq \underline{\underline{10\,m}}$$

5

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 12, 2007
11:10 AM – 1:10 PM

General Physics (Part II)
Section 6.

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Good luck!!

1. Hawking showed a black hole has an entropy $S = \beta k_B A$, where A is the surface area of the black hole, k_B is Boltzmann's constant, and β is a factor of the form $\beta = \alpha \hbar^m G^n c^p$. Here α is a numeric factor of order unity, $\hbar$ is Planck's constant, G is the gravitational constant, c is the speed of light, and m, n, p are integers. The radius of a black hole can be expressed as $R = 2GM / c^2$, where M is the mass of the black hole.

Develop a semi-classical argument for black hole evaporation.

(a) Determine m, n, and p and thus β purely on the basis of dimensional arguments.

(b) Find an explicit expression for the entropy as a function of mass M and other constants, i.e. $S = S(M)$

(c) Hawking's First Law of Black Hole Thermodynamics relates the rest mass energy of the black hole to its temperature T and entropy S by the relation $dE = TdS$. Derive an explicit expression for the temperature of the black hole as a function of mass, i.e. $T = T(M)$.

(d) Hawking showed the black hole radiates away its rest mass energy as a perfect black body radiator. Find an explicit expression for how long it take the black hole to radiate away its mass. Given a lifetime τ, your expression should be of the form $\tau = \tau(M)$.

2. The Earth has a thin atmosphere (thin compared to the Earth's radius). Assuming the atmosphere is isothermal (uniform temperature), show that the number density of air molecules as a function of atmospheric height should have an exponential form: $n(h) \propto \exp(-h / h_0)$, where h is the height above the Earth's crust. Express the scale height h_0 in terms of the relevant physical parameters (such as temperature and so on), and provide a numerical estimate.

Useful facts:
The gravitational acceleration at the Earth's surface is $g = 9.8\mathrm{m/s^2}$.
The Boltzmann constant is $k_B = 1.3807\mathrm{x}10^{-23}$ Joules/K
The proton mass is: $m_p = 1.6726\mathrm{x}10^{-27}$ kg

3. A wire of length L and 1-dimensional mass density μ is fixed at both ends along the x direction and is tightened so that there is a tension τ along the wire. Small displacements y along the direction perpendicular to the wire satisfy the equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\mu}\frac{\partial^2 y}{\partial x^2}.$$

(a) Show that the total energy per normal mode is given by

$$E_n = \frac{\tau \pi^2 n^2}{4L} A_n^2$$

where A_n is the amplitude of the n^{th} normal mode.

(b) What is the RMS fluctuation of y at the midpoint of the wire, when it is in equilibrium with a heat bath at temperature T? Neglect the contribution of zero-point motion.

A useful series is $\sum_{m=0}^{\infty}(2m+1)^{-2} = \frac{\pi^2}{8}$.

4. An experimental chamber is equipped with a diamond window. You are planning to use a green laser light that enters this chamber perpendicularly to this window. What material would you use to create a single thin film anti-reflection coating (ARC) at the air-to-diamond face? How thick is the proposed layer? How good of an ARC is it?

a) Please derive the general condition for ARC as the function of the index of refraction of the three materials.

b) Please provide the general condition for the film's thickness.

c) Using the data below, determine the best coating material for the film.

d) Discuss the residual reflectivity of the coated diamond. Which is easier to deal with: a diamond or a glass window?

(*Index of refraction of various materials:* Teflon 1.35 - 1.38, Glycerol 1.4729, Acrylic glass 1.490 - 1.492, Rock salt 1.516, Crown glass (pure) 1.50 - 1.54, Salt (NaCl) 1.544, Polycarbonate 1.584 - 1.586, Flint glass (pure) 1.60 - 1.62, Crown glass (impure) 1.485 - 1.755, Magnesium Fluoride (MgF_2) 1.38, Bromine 1.661, Flint glass (impure) 1.523 - 1.925, Cubic Zirconia 2.15 - 2.18, Diamond 2.419, Moissanite 2.65 - 2.69, Cinnabar (Mercury sulfide) 3.02, Gallium(III) phosphide 3.5, Gallium(III) arsenide 3.927 Silicon 4.01)

5. The following problem does not have a precise answer, but it has a range of answers that most scientists and engineers would agree upon. More importantly, the answer is derived from a set of (reasonable) assumptions and follows a logical path that most people trained in the sciences would pursue. This kind of thinking is what is tested here.

(a) How many photons (in the visible spectrum) are in your qualifying exam room right now? Present the string of estimates and arguments by which you arrive at your number. (We assume the room has no windows and is illuminated by electrical lighting. If this is not the case imagine it to be such a room.)

(b) Compare this number with the number of gas molecules in your qualifying exam room.

(c) Replace the walls and ceiling, but not the floor with mirrors that are 99% reflective. Will the number of photons in your room change? If yes, by how much? Is it significant, given the approximations you have made?

(d) If you wanted to increase the number of photons by a factor of 10 what could you do? Make at least 2 different proposals.

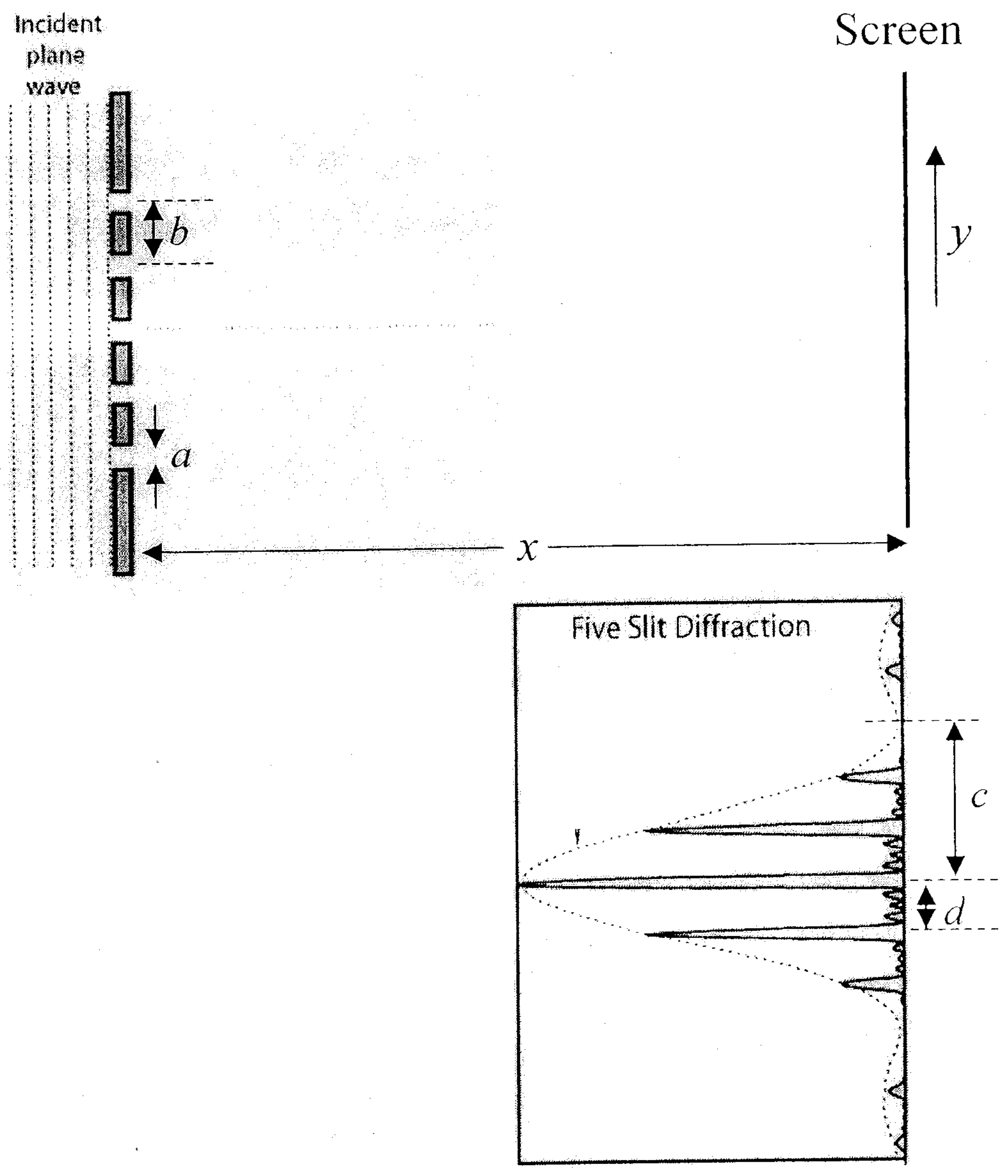
6. The attached figure shows the Fraunhofer diffraction pattern for a five slit pattern. Assume that x is much larger than a and b. Assume the wavelength of the incident light to be λ, which is much smaller than a. a is significantly smaller than b.

(a) Obtain the interval d between the adjacent sharp strong peaks on the screen.

(b) Explain the reason for reduction of peak intensity shown by the dotted line of the intensity envelope

(c) Obtain the distance c on the screen where the intensity becomes zero.

(d) Between adjacent strong intensity peaks, we see three very weak small intensity peaks. Explain the reason for this phenomenon. Why do we see N-2 such small peaks, where N=5 is the number of the slits?



NOV 28 2006

Hawking showed a black hole has an entropy $S = \beta K A$ where A is the surface area of the black hole, K is Boltzmann's constant and β is a factor of the form $\beta = \alpha \hbar^m G^n c^p$ where α is a numeric factor of order unity and m, n, p are integers. The radius of a black hole can be expressed as $R = \frac{2GM}{c^2}$, where M is the mass of the black hole.
Develop a semi-classical argument for black hole evaporation.

a.) Determine m, n, p and thus β purely on the basis of dimensional arguments.

b.) Find an explicit expression for the entropy as a function of mass M and other constants, ie $S = S(M)$.

c.) Hawking's 1st Law of Black Hole Thermodynamics relates the rest mass energy of the black hole to its temperature T and entropy S by the relation $dE = TdS$. Derive an explicit expression for the temperature of the black hole as a function of mass ie $T = T(M)$.

d.) Hawking showed the black hole radiates away its rest mass energy as a perfect black body radiator. Find an explicit expression for how long it takes the black hole to radiate away its mass. Given a lifetime τ your expression should be of the form $\tau = \tau(M)$

e.) Based on your answer in (d) do you expect primordial black holes ($M \sim 10^{15}$ g) or "stellar" black holes ($M \sim 10^{34}$ g) to live longer?

Modern: solution; Hailey

a.) The dimensions of entropy are K, so βA must be dimensionless or

$$L^2 \hbar^m G^n c^p \sim \text{dimensionless}$$

$$L^2 \left(\frac{ML^2}{T}\right)^m \left(\frac{L^3}{MT^2}\right)^n \left(\frac{L}{T}\right)^p \sim \text{dimensionless}$$

$$2 + 2m + 3n + p = 0 \qquad (L)$$

$$m - n = 0 \qquad (M)$$

$$-m - p - 2n = 0 \qquad (T)$$

This trivially solves to

$$m = n = -1 \ ; \ p = 3 \qquad \underset{\sim}{\text{Ans}}$$

$$S = \frac{c^3}{G\hbar} \alpha A K \qquad \underset{\sim}{\text{Ans}}$$

b.) $$S = \frac{\alpha K c^3}{G\hbar} 4\pi \left(\frac{2GM}{c^2}\right)^2$$

$$S = \frac{16\pi G \alpha K M^2}{c\hbar} \qquad \underset{\sim}{\text{Ans}}$$

3/4

~~2/4~~

c.) $$dE = c^2\, dM = T\, d\left(\frac{16\pi G \alpha K M^2}{\hbar c}\right)$$

$$c^2\, dM = T \cdot \frac{32\pi G \alpha K M\, dM}{\hbar c}$$

$$T = \frac{\hbar c^3}{32\pi \alpha K G}\,\frac{1}{M}$$ Ans

d.) $$dE/dt = -4\pi R^2 \sigma T^4$$ is the radiation of a black body. Here

$E = Mc^2$, $R = \frac{2GM}{c^2}$ $\Rightarrow$

$$c^2 \frac{dM}{dt} = -4\pi \left(\frac{2GM}{c^2}\right)^2 \left(\frac{\hbar c^3}{32\pi \alpha K G}\right)^4 \frac{1}{M^4}$$

We see at once that

$$\tau \equiv \int_0^{\tau} dt = -\Gamma \int_{M_h}^{0} M^2\, dM$$

where Γ is a constant or

$$\tau \propto M^3.$$ Ans

e.) Clearly a stellar mass black hole will last much longer ie will take much longer to evaporate.

~~3/4~~ 4/4

Hui Sect 6 9lr II #2

General -

Note: the following problem assumes that the students are provided a list of fundamental constants, including the Boltzmann constant and the proton mass. If that is not true, these numbers should be provided for the following problem.

Problem:

The Earth has a thin atmosphere (thin compared to the Earth's radius). Assuming the atmosphere is isothermal (uniform temperature), show that the number density of air molecules as a function of atmospheric height should have an exponential form: $n(h) \propto \exp[-h/h_0]$, where h is the height above the Earth's crust. Express the scale height h_0 in terms of the relevant physical parameters (such as temperature and so on), and provide a numerical estimate. You might find it useful to know that the gravitational acceleration at the Earth's surface is $9.8\,\mathrm{m/s^2}$.

Solution:

Balancing pressure gradient and gravity, one has $dP/dh = -mng$ where m is the mean molecular mass and g is the gravitational acceleration. Adopting $P = nkT$, and assuming T is a constant, we have $dn/dh = -mgn/(kT)$, which can be integrated to give $n(h) \propto \exp[-h/h_0]$, with $h_0 = kT/(mg)$. A reasonable T to assume is the freezing point $T \sim 273\mathrm{K}$. A reasonable mass would be $30m_p$, where m_p is the mass of the proton. Recall that the nitrogen molecule weighs about $28m_p$ and the oxygen molecule weighs about $32m_p$. Therefore $h_0 = (1.38 \times 10^{-23} \times 273)/(30 \times 1.67 \times 10^{-27} \times 9.8)$ m which is about 8 km. The actual scale height of our atmosphere is about 6 km, so this is not a bad estimate. Incidentally, the tallest mountains are about 8 km. A coincidence...

(a) For the nth order normal mode,

$$y_n(x,t) = A_n \sin\left(\frac{\pi}{L} n x\right) \sin(\omega_n t + \phi)$$

ω_n can be obtained from $\frac{\partial^2 y_n}{\partial t^2} = \frac{\tau}{\mu}\frac{\partial^2 y}{\partial x^2}$

$$\Rightarrow \quad \omega_n = \sqrt{\frac{\tau}{\mu}}\left(\frac{\pi}{L} n\right)$$

Kinetic energy of string

$$K_n = \frac{1}{2}\int_0^L dx\, \mu \underbrace{\left[\int_0^T \left(\frac{\partial y_n}{\partial t}\right)^2 dt\right]\frac{1}{T}}_{A_n^2 \omega_n^2 \cdot \frac{1}{2}} \qquad T = \frac{2\pi}{\omega_n}$$

$$= \frac{1}{4}\mu \frac{\tau}{\mu}\left(\frac{\pi}{L} n\right)^2 A_n^2 \underbrace{\int_0^L \sin^2\left(\frac{\pi}{L} n x\right) dx}_{= \frac{1}{2}L}$$

$$= \frac{1}{8}\frac{\tau \pi^2 n^2}{L} A_n^2$$

$$E_n = 2K_n = \underline{\frac{1}{4}\frac{\tau \pi^2 n^2}{L} A_n^2}$$

(b) From equipartition theorem,

$$k_B T = E_n = \frac{1}{4}\frac{\tau\pi^2 n^2}{L} A_n^2 \Rightarrow A_n^2 = \frac{4 L k_B T}{\tau \pi^2 n^2}$$

$$y\left(x = \frac{L}{2}\right) = \sum_{n:\text{odd}} A_n^2 \sin^2(\omega_n t + \phi)$$

$$\langle \delta y^2 \rangle = \frac{4 k_B T L}{\tau \pi^2} \sum_{n:\text{odd}} \frac{1}{n^2} \underbrace{\langle \sin^2(\omega t + \phi)\rangle}_{= \frac{1}{2}}$$

$$= \frac{2 k_B T L}{\tau \pi^2}\frac{\pi^2}{8} \quad \text{or} \quad \underline{y_{rms} = \sqrt{\langle \delta y^2 \rangle} = \frac{1}{2}\sqrt{\frac{k_B T L}{\tau}}}$$

Sec 6 q in II #5

Horst Stormer 2007 Quals
12/7/06 General

HOW MANY PHOTONS ARE IN THIS ROOM?

The following problem does not have a precise answer, but it has a range of answers that most scientists and engineers would agree upon. More importantly, the answer is derived from a set of (reasonable) assumptions and follows a logical path that most people trained in the sciences would pursue. This kind of thinking is what is tested here.

A) (7 points)
How many photons (in the visible spectrum) are in your qualifying exam room right now? Present the string of estimates and arguments by which you arrive at your number. (We assume the room has no windows and is illuminated by electrical lighting. If this is not the case imagine it to be such a room.)

B) (2 point)
Compare this number with the number of gas molecules in your qualifying exam room.

C) (4 points total)
Replace the walls and ceiling, but not the floor with mirrors that are 99% reflective. Will the number of photons in your room change? (1 point) If yes, by how much? (2 points) Is it significant, given the approximations you have made? (1 point)

D) (2 points total)
If you wanted to increase the number of photons by a factor of 10 what could you do? Make at least 2 different proposals. (1 point each)

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

SOLUTIONS:

A) Count the number of neon lights and/or light bulbs in the room. A neon light uses typically 50W, a light bulb typically 100W of electrical power. The efficiency (Watts of visible light per Watts of electrical power) may be ~20% for neon lights and 10% for incadescant lights. Hence both type of bulbs create about 10W of visible light. Let's assume you counted about 100 bulbs in your room. This makes for 1kW of visible light. With ~1eV of energy per optical photon this means that
$1000W/1eV \sim 1000W/10^{-19}AsecV = 10^{22}$photons/sec are being created. Now one has to determine the lifetime of a photon. Photons are being absorbed by the walls and objects in the room. One may guess that, the reflectivity is about 30%, i.e. a photon makes it about 3 times across the room. Assuming the room is about 10m x 10m x 5m one may assume a 30m mean free path, which translates into a lifetime of $30m/3x10^8 m/sec = 10^{-7}$sec. This means $10^{22} \times 10^{-7} = 10^{15}$ photons are in this room at any given moment.

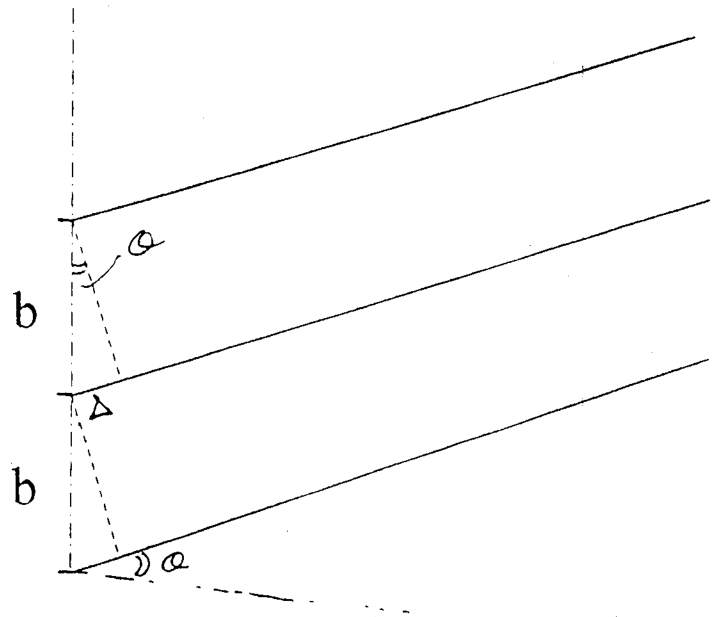
B) One mole is 6×10^{23} molecules and fills 22.3 l at ambient conditions. The room has a volume of $10 \times 10 \times 5 \sim 500 m^3$ equivalent to 5×10^5 l/(22 l) ~ 3×10^4 mole, equivalent to 2×10^{28} molecules. Hence, there are many more molecules than photons in this room.

C) If you were to replace the walls and ceiling, but not the floor, with highly reflecting mirrors the mean free path of the photons would extend, but not by very much. One could argue that you have replaced ~70% of the inner surface area of the room with ideal reflectors, which would increase the mean free path of a photon by about a factor of 1/(1-.7)~3. Hence the number of surviving photons is ~3 times higher at any given instant. This is not very significant given all of the assumptions that have been made.

D) One can increase the number photons in the room by two means. Increase the creation rate (10 times more light bulbs) or increase the lifetime by a factor of 10, by increasing the reflectivity of all objects in the room, such as painting everything white, covering it with aluminum foil, placing mirrors on the walls and floors.

Fraunhofer key Tomo Uemura JAN 12 2007 Sec 6 #6 solution

(1)



light path difference Δ

$b \sin\theta = \Delta = n\lambda$

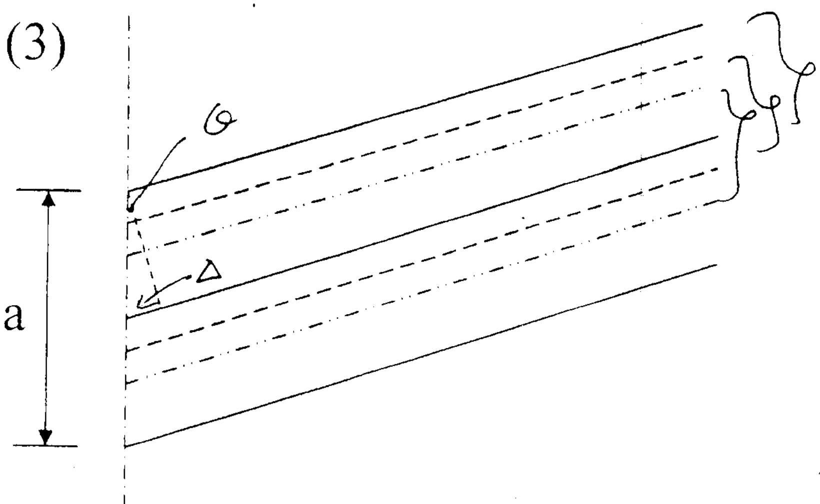
$x \tan\theta = d$

$\theta = \frac{\lambda}{b}$ $\left\{ \begin{array}{l} \theta \ll 1 \\ n = 1 \end{array} \right.$

$\therefore \; d = x \cdot \frac{\lambda}{b}$

(2) envelope is due to interference of wave fronts passing through <u>each</u> hole. (i.e., single-slit envelope).

(3)

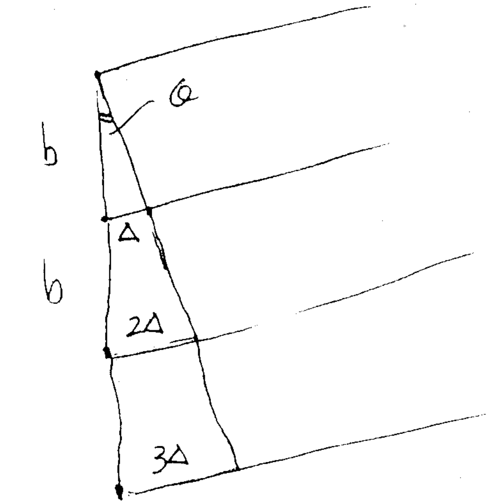


(3) for each set of wavefront passing distance of $a/2$, in negative interference, the intensity becomes 0.

$\frac{a}{2} \cdot \sin\theta = \frac{n}{2}\lambda$ $x \tan\theta = c$

$\therefore \; c = x\theta = x \cdot \frac{\lambda}{a}$

(4).



$$k \equiv \frac{2\pi}{\lambda}$$

$$b\theta = \Delta$$

$\Delta k \longrightarrow$ phase change for adjacent holes.

$$\Delta k \equiv \phi$$

intensity I

N holes

$$I \propto \left| 1 + e^{i\phi} + e^{2i\phi} + \cdots e^{(N-1)i\phi} \right|^2$$

$$= \left| \frac{1 - e^{iN\phi}}{1 - e^{i\phi}} \right|^2.$$

Numerator becomes 0

for $N\phi = 2\pi \times m$ ← integer

$m = 0$ and N correspond to main peaks.

$\Rightarrow$ $N-1$ zeros between main peaks

$\Rightarrow$ $N-2$ sub peaks.