The Fundamental Theorem of Riemannian Geometry

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Abstract

This article is an introduction to affine and Riemannian connections on manifolds. In particular, we will prove the Fundamental Theorem of Riemannian Geometry: every Riemannian manifold has a unique Riemannian connection.

Audience: This article assumes an introductory knowledge of inner products, manifolds, directional derivatives, and vector fields as taught in this course MATH 4081.

1 Affine Connections

On a manifold M, we know how to define the directional derivative of a smooth function f in the direction $X_p \in T_pM$:

$$\nabla_{X_p} f = X_p f$$

It is also useful to talk about directional derivatives of vector fields on manifolds. However, there is no canonical way to define the directional derivative of a vector field since there is no canonical basis for the tangent space T_pM. Instead, we define a map called an affine connection with similar linearity and Leibniz Rule properties as directional derivatives in $\mathbb{R}^n$. This affine connection acts like a directional derivative of vector fields on a manifold.

Definition (Affine Connection)**.** *Let $\mathfrak{X}(M)$ be the set of all C^∞ vector fields on M. An affine connection on a manifold M is an $\mathbb{R}$-bilinear map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

satisfying the two properties: If $\mathcal{F}$ is $C^\infty(M)$, the set of smooth functions on M, then

1. *$\nabla(X, Y)$ is $\mathcal{F}$-linear in X*
2. *$\nabla(X, Y)$ satisfies the Leibniz Rule in Y, i.e. for $f \in \mathcal{F}$, $\nabla(X, fY) = (Xf)Y + f\nabla(X, Y)$*

Following convention , we write $\nabla_X(Y)$ for $\nabla(X, Y)$.

Example 1.1 (Directional Derivative)**.** The directional derivative D_X of a vector field Y on $\mathbb{R}^n$ is an affine connection on $\mathbb{R}^n$. It is called the Euclidean connection.

Example 1.2 (A Connection on S^2). Let dY be the differential (Jacobian matrix) of a vector field $Y : S^2 \to R^3$ such that $\langle Y(p), p\rangle = 0$ for all $p \in S^2$. Then,

$$\nabla_X Y(p) = dY(p)(X(p)) + \langle X(p), Y(p)\rangle p$$

defines an affine connection on S^2, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^3$.

2 Torsion

An interesting property of affine connections on manifolds is torsion. For our purposes, torsion will be useful insofar as it is 0 for a particular connection, the Riemannian connection, in section 3.2.

Definition (Torsion)**.** *The torsion tensor or torsion T of a connection ∇ is the quantity*

$$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y]$$

If $T(X,Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$, then we say the associated connection is **torsion-free**.

Example 2.1. The connection in Example 1.2 on S^2 is torsion-free.

An interesting property of torsion is that it is $\mathcal{F}$-linear in both arguments (as shown below) even though $\nabla_X Y$ is not necessarily $\mathcal{F}$-linear in Y.

Proposition 2.1. *Torsion $T(X,Y)$ of a connection ∇ is $\mathcal{F}$-linear in X and Y.*

Proof. Let $f, g \in \mathcal{F}$. Then,

$$\begin{aligned}
T(fX, gY) &= \nabla_{fX}(gY) - \nabla_{gY} fX - [fX, gY] \\
&= f\nabla_X(gY) - g\nabla_Y(fX) - [fX, gY] \text{ ($\mathcal{F}$-linearity in first argument)} \\
&= f(Xg)Y + fg\nabla_X(Y) - g(Yf)X - gf\nabla_Y(X) - [fX, gY] \text{ (Leibniz Rule)} \\
&= f(Xg)Y + fg\nabla_X(Y) - g(Yf)X - fg\nabla_Y(X) - f(Xg)Y + g(Yf)X - fg[X,Y] \\
&= fg(\nabla_X(Y) - \nabla_Y(X) - [X,Y] \\
&= fgT(X,Y)
\end{aligned}$$

□

3 Riemannian Connections

3.1 Riemannian Manifolds

We continue to build up our machinery for proving the fundamental theorem of Riemannian geometry. A topic central to Riemannian geometry is Riemannian manifolds. One can think of a Riemannian manifold as a manifolds equipped with a metric that allows one to measure lengths and angles of tangent vectors intrinsically to the manifold, independent of its embedding in a higher dimensional space.

Definition (Riemannian Metric)**.** *A Riemannian metric on a manifold M is the assignment of an inner product $\langle \cdot, \cdot \rangle_p$ on the tangent space T_pM to each $p \in M$ such that $p \mapsto \langle \cdot, \cdot \rangle_p$ is C^∞ as follows: If X, Y are C^∞ vector fields on M, then $p \mapsto \langle X_p, Y_p \rangle_p$ is a C^∞ function on M.*

Example 3.1 (Euclidean Metric on $\mathbb{R}^n$)**.** All tangent spaces $T_p\mathbb{R}^n$ are canonically isomorphic to R^n. So, the standard Euclidean inner product $\langle \cdot, \cdot \rangle$ on R^n defines a Riemannian metric on R^n. This is called the Euclidean metric on $\mathbb{R}^n$.

Definition (Riemannian Manifold)**.** *A Riemannian manifold is a pair $(M, \langle \cdot, \cdot \rangle)$ of a manifold M and a Riemannian metric $\langle \cdot, \cdot \rangle$ on M.*

Example 3.2. $\mathbb{R}^n$ equipped with the Euclidean metric is a Riemannian manifold.

Example 3.3. S^2 equipped with the Euclidean metric on R^3 is a Riemannian manifold.

Remark. Although we won't prove it, we should note that on every manifold M, there is a Riemannian metric.

Proposition 3.1. *A C^∞ vector field X on a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is uniquely determined by the values $\langle X, Z\rangle$ for all $Z \in \mathfrak{X}(M)$.*

Proof. Let $X, X' \in \mathfrak{X}(M)$. We want to show that if $\langle X, Z\rangle = \langle X', Z\rangle$ for all $Z \in \mathfrak{X}(M)$, then $X = X'$. This is equivalent to showing that for $Y = X - X'$, if $\langle Y, Z\rangle = 0$ for all $Z \in \mathfrak{X}(M)$, then $Y = 0$.

Choose $Z = Y$. Then,

$$\begin{aligned}\langle Y, Z\rangle = \langle Y, Y\rangle = 0 &\implies \langle Y_p, Y_p\rangle_p = 0 \text{ for all } p \in M \\ &\implies Y_p = 0 \text{ for all } p \in M \\ &\implies Y = 0\end{aligned}$$

Thus, $X = X'$ □

3.2 Riemannian Connections

On an arbitrary (Riemannian) manifold, we may define a multitude of connections. However, if we wanted to define unique connections on Riemannian manifolds, we would need to impose restrictions on the set of connections on our manifold. In fact one can find a unique "Riemannian connection" on every Riemannian manifold by imposing just two restrictions on our set of connections: metric compatibility and torsion-freeness.

Definition (Metric Compatibility of a Connection)**.** *On a Riemannian manifold M, a connection is compatible with the metric if for all $X, Y, Z \in \mathfrak{X}(M)$,*

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

Definition (Riemannian Connection)**.** *On a manifold, a Riemannian connection is a connection that is torsion-free and is compatible with the metric.*

Example 3.4. The Euclidean connection in Example 1.1 on $\mathbb{R}^n$ equipped with the Euclidean metric is a Riemannian connection.

Example 3.5 (Riemannian Connection on S^2)**.** The affine connection in Example 1.2 is a Riemannian connection on S^2. We already know it is torsion-free by Example 2.1. One can show that it is metric compatible as well.

Note that the form of the Riemannian connection on S^2 in Example 1.2 is interesting. It happens to be the projection of $D_{X_p}Y$, the directional derivative of Y in the X_p-direction, onto T_pS^2. More generally, the point-wise Riemannian connection for a smooth, not necessarily orientable, surface M in $\mathbb{R}^3$ is given by

$$(\nabla_X Y)_p = pr_p(D_{X_p}Y)$$

where $pr_p : T_p\mathbb{R}^3 \to T_pM$ is the projection to the tangent space of M at p.

Now that we have built up this machinery of connections, torsion, metric compatibility, and Riemannian manifolds & connections, we are prepared to prove the following theorem:

Theorem 3.1 (Fundamental Theorem of Riemannian Geometry)**.** *On a Riemannian manifold M, there is a unique Riemannian connection.*

Proof. Suppose ∇ is a Riemannian connection on M. By Proposition 3.1, to specify $\nabla_X Y$, it suffices to know $\langle \nabla_X Y, Z\rangle$ for all $Z \in \mathfrak{X}(M)$.

Since ∇ is a Riemannian connection, it is metric compatible. So, permuting cyclically in X, Y, Z,

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$$
$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Since ∇ is a Riemannian connection, it is also torsion-free. So, for all $X, Y \in \mathfrak{X}(M)$,

$$T(X, Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y] = 0$$

So, we may rewrite $Y\langle Z, X\rangle$ as

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle - \langle Z, [X, Y]\rangle$$

So, we may write

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle &= 2\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle + \langle X, \nabla_Y Z - \nabla_Z Y\rangle - \langle Z, [X, Y]\rangle \\ &= 2\langle \nabla_X Y, Z\rangle + \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle - \langle Z, [X, Y]\rangle \text{ (torsion-free)} \end{aligned}$$

Thus,

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\big(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle\big)$$

By Proposition 3.1, this defines $\nabla_X Y$ for all $X, Y \in \mathfrak{X}(M)$. So, if a Riemannian connection ∇ exists, it is unique and defined as above.

We may also see that a connection ∇ defined as above is torsion-free and compatible with the metric. Torsion-freeness may be proved by calculating $\langle T(X, Y), Z\rangle$ and showing that it is 0 for all $X, Y, Z \in \mathfrak{X}(M)$. So, $T(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$. Metric compatibility may be proved by simply calculating $\langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$ and showing it is equal to $Z\langle X, Y\rangle$. Thus, ∇ is a Riemannian connection. □

References

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