# The Dirac-von Neumann and Hilbert Space Formulations of Quantum Mechanics

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**Abstract**

Our goal in this article is to construct and understand the mathematics behind the Dirac-von Neumann Operator Algebra formulation and the Hilbert Space formulation of Quantum Mechanics. Along the way, we will explore topological vectors spaces like Banach and Hilbert spaces, continuous dual spaces, weak and weak$^\star$ topologies, and C* algebras, in the hopes of understanding the mathematical foundations of Quantum Physics.

This article assumes a working intuition/understanding of real analysis, linear algebra, and point-set topology. Some familiarity with quantum mechanics will be useful, but is not necessary.

## Contents

**Definition.** *A Topological Vector Space (TVS) $X$ is a vector space over a topological field $\mathbb{K}$ (usually $\mathbb{R}_{std}$ or $\mathbb{C}_{std}$) that is equipped with some topology such that addition $+ : X \times X \to X$ and multiplication $\cdot : \mathbb{K} \times X \to X$ are continuous.*

# 1 Banach and Hilbert Spaces

## 1.1 Banach Spaces

We begin our journey with an exploration of Banach spaces. Banach spaces are of interest in various areas of mathematics including functional analysis, partial differential equations, and manifolds. Specific Banach spaces like Hilbert spaces are of fundamental importance to physics including quantum mechanics, topological and algebraic quantum field theory, and classical field theory.

**Definition.** *A Banach space $B$ is a normed complete TVS. A metric space $(X, d)$ is complete is every Cauchy sequence of points in $X$ also has a limit in $X$. A sequence of points $x_1, x_2, ..., x_n \in X$ is Cauchy if for all $\epsilon > 0$, there is some $N$ such that $d(x_m, x_n) < \epsilon$ for all $m, n > N$. In a Banach space, the metric is given by its norm $\|\cdot\|$.*

**Example 1.1.** An example of a Banach space (or really, multiple examples) is $\ell^p$ for $1 \leq p < \infty$, the space of all complex sequences $x = \{x_i\}$ such that

$$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p} < \infty$$

Another example of a closely related Banach space is $\ell^\infty$, the space of all complex sequences $x = \{x_i\}$ such that

$$\|x\|_\infty = \sup(|x_i|) < \infty$$

Although not immediately relevant, it is important for us to consider closed subspaces of Banach spaces. We will later see the following Proposition come into action in Section 3.

**Proposition 1.1.** *A closed subspace $C \subseteq B$ of a Banach space $B$ is itself a Banach space.*

*Proof.* Since $C$ is a subspace, it is vector space with norm inherited from $B$. Given any Cauchy sequence $\{x_i\}$ in $C$, it must have a limit in $B$ since $B$ is complete. Since $C$ is closed, it must contain all limit points - so the limit of $\{x_i\}$ is also contained in $C$. So, $C$ is also complete. Thus, $C$ is a complete normed TVS - a Banach space. □

A natural quantity to examine on Banach spaces are maps between Banach spaces. These maps themselves have a norm as defined below - so a set of maps between Banach spaces can form a Banach space itself.

**Definition.** *The operator norm for an map $f : X \to Y$ between Banach spaces $X, Y$ is $\|f\| = \sup_{\|x\|\leq 1}(\|f(x)\|)$. We say $f$ is bounded if there exists a constant $M > 0$ such that for all $x \in X$, $\|f(x)\| \leq M\|x\|$.*

**Proposition 1.2.** *A linear map $f : X \to Y$ between Banach spaces is continuous if and only if it is bounded.*

*Proof.* First, suppose $f : X \to Y$ is bounded, i.e. there exists a constant $M > 0$ such that for all $x \in X$, $\|f(x)\| \leq M\|x\|$. Then, for all $x, y \in X$,

$$\|f(x) - f(y)\| = \|f(x - y)\| \leq M\|x - y\|$$

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M}$. Then, $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$. Thus, $f$ is continuous.

Now, suppose $f$ is continuous - specifically continuous at $0 \in X$. Since $f$ is linear, $f(0) = 0$. From the $\epsilon$-$\delta$ definition of continuity, take $\epsilon = 1$. Then, there is a $\delta$ such that

$$\|f(x)\| = \|f(x) - f(0)\| \leq 1 \text{ whenever } \|x\| = \|x - 0\| \leq \delta$$

So, $\|f(x)\| = \frac{\|x\|}{\delta}\|f(\delta\frac{x}{\|x\|})\|$. Note that since $\|\delta\frac{x}{\|x\|}\| \leq \delta$, $\|f(\delta\frac{x}{\|x\|})\| \leq 1$. Thus,

$$\|f(x)\| \leq \frac{\|x\|}{\delta}$$

Thus, $f$ is bounded by $M = \frac{1}{\delta}$. □

We denote the set of continuous linear maps from a Banach space $X$ to another Banach space $Y$ as $CL(X, Y)$.

**Proposition 1.3.** *$CL(X, Y)$ for $X$ and $Y$ Banach spaces is itself a Banach space when equipped with the operator norm.*

*Proof.* We have to show that $CL(X, Y)$ is a normed TVS that is complete.

*Normed TVS:* From linear algebra, we know that the set of all linear maps between $L(X, Y)$ between vector spaces $X$ and $Y$ is itself a vector space. Then, if $CL(X, Y)$ is a subset of $L(X, Y)$ closed under addition and scalar multiplication, then it is also a vector space. Let $a, b \in \mathbb{K}$ and $f, g \in CL(X, Y)$ with $\|f\| = M$, $\|g\| = N$. Then,

$$\|af + bg\| = \sup_{\|x\| \leq 1}(\|af(x) + bg(x)\|) \leq \sup_{\|x\| \leq 1}(|a|\|f(x)\| + |b|\|g(x)\|) = |a|M + |b|N$$

So, $CL(X, Y)$ is closed under addition and scalar multiplication - it is a normed vector space.

*Completeness:* We now show that $CL(X, Y)$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $CL(X, Y)$. Since $Y$ is complete, $\{f_n(x)\}$ is Cauchy in $Y$ and converges uniformly to some $f(x) \in Y$ for all $x \in X$. Given $\epsilon > 0$, choose $N > 0$ so that for all $m, n > N$,

$$d(f_m(x), f_n(x)) = \|f_m(x) - f_n(x)\| < \epsilon$$

Then, if we take $m \to \infty$, we have for all $x \in X$ and $n > N$,

$$\|f(x) - f(x_n)\| < \epsilon$$

So, $f_n \to f$ uniformly. We just need to show that $f$ is bounded and therefore lies in $CL(X, Y)$. By uniform convergence, there exists some $N > 0$ such that for all $n > N$, $x \in X$,

$$\|f(x) - f_n(x)\| \leq 1$$

Since $f_n \in CL(X, Y)$ are each bounded, $M > 0$ such that $\|f_n\| < M - 1$. Then, for all $\|x\|_X \leq 1$,

$$\|f(x)\| \leq \|f(x) - f_n(x)\| + \|f_n(x)\| \leq 1 + M - 1 = M$$

So, $f$ is bounded and is in $CL(X, Y)$. Thus, $CL(X, Y)$ is a complete normed TVS, i.e. Banach. □

## 1.2 Hilbert Spaces

We now turn to a specific type of Banach spaces called Hilbert spaces. Quantum states and physical observables that we can measure are intimately tied to Hilbert spaces and operators between them. So, understanding Hilbert spaces is essential to understanding the mathematical foundations of quantum physics.

**Definition.** *A Hilbert space is a pair* $(H, \langle\cdot,\cdot\rangle)$ *of a Banach space* $H$ *and an inner product* $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{K}$ *so that the associated norm is defined by* $\|x\| = \sqrt{\langle x, x\rangle}$.

The inner product on a Hilbert space is a more primitive notion than the norm - we generally don't talk about the norm on a Hilbert space, but instead of the inner product on it. For our purposes, an inner product on a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and a Hilbert space $H$ satisfies the following four conditions:

1. Linear in first coordinate, i.e. $\langle a(x+y), z\rangle = a(\langle x, z\rangle + \langle y, z\rangle)$ for $a, b \in \mathbb{K}$ and $x, y, z \in H$
2. Antilinear in second coordinate, i.e.$\langle x, b(z+y)\rangle = \overline{b}(\langle x, z\rangle + \langle x, y\rangle)$, where $\overline{b}$ is the complex conjugate of $b$
3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
4. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0 \in H$

**Example 1.2.** An example of a Hilbert space is $\ell^2$ when equipped with the inner product $\langle u, v\rangle = \sum u_i \overline{v}_i$ for $u = \{u_i\}, v = \{v_i\}$, where $\bar{\cdot} : \mathbb{K} \to \mathbb{K}$ is complex conjugation.

It is important to note that not all Banach spaces are Hilbert spaces. In fact, Hilbert spaces are somewhat rare amongst Banach spaces, as we will understand by this next Proposition.

**Proposition 1.4.** ***(Parallelogram Law)*** *A Banach space* $B$ *with norm* $\|\cdot\|$ *is a Hilbert space if and only if it satisfies the parallelogram law, i.e. for all* $u, v \in B$, $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$.

*Proof.* Suppose $B$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Then, for arbitrary $u, v \in B$,

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle + \langle u, u\rangle + \langle u, -v\rangle + \langle -v, u\rangle + \langle -v, -v\rangle \\
&= 2\langle u, u\rangle + 2\langle v, v\rangle \\
&= 2(\|u\|^2 + \|v\|^2)
\end{aligned}$$

Conversely, suppose $u, v \in B$, $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$ for all $u, v \in B$, a Banach space. Then, we may define an inner product $\langle\cdot,\cdot\rangle : B \times B \to \mathbb{K}$ by

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2)$$

One can check that this is a valid inner product that is consistent with the parallelogram law. Equipped with this inner product, B is a Hilbert space. □

Similar to our discussion on Banach spaces, let us observe something important about the set of continuous linear maps on Hilbert spaces:

**Proposition 1.5.** *The set of continuous linear functions $CL(X, Y)$ between Hilbert spaces $X$ and $Y$ is a Banach space, but not necessarily a Hilbert space.*

*Proof.* Since $X, Y$ are Hilbert and so Banach, Proposition 1.3 tells us that $CL(X, Y)$ is a Banach space. To show that $CL(X, Y)$ is not always Hilbert, let's focus on $CL(X, X) = CL(X)$, the set of bounded operators on $X$. The reason we're examining this space is because it will be useful to us in Section 3.

Suppose $\dim(X) > 1$. Then, we can find $y, z \in X$ with $\|x\| = \|y\| = 1$ and $\langle y, z\rangle = 0$. Define projection operators $P_y, P_z \in CL(X)$ such that $P_y(x) = \langle x, y\rangle y$ and $P_z(x) = \langle x, z\rangle z$. One can check that $P_y, P_z$ are continuous and linear on $X$. Then,

$$\|P_y\| = \sup_{\|x\|\leq 1}(\|P_y(x)\|) = 1 \text{ and } \|P_z\| = \sup_{\|x\|\leq 1}(\|P_z(x)\|) = 1$$

So, $2(\|P_y\|^2 + \|P_z\|^2) = 4$. One can also check that $\|\langle x, y\rangle y \pm \langle x, z\rangle z\|^2 = \|\langle x, y\rangle\|^2 + \|\langle x, z\rangle\|^2$ because $\langle y, z\rangle = 0$ and $\|y\| = \|z\| = 1$. Since $\langle y, z\rangle = 0$, we may think of $P_y(x)$ and $P_z(x)$ as vectors in the $y$-$z$ plane. Then, for $\|x\| \leq 1$, $\|\langle x, y\rangle\|^2 + \|\langle x, z\rangle\|^2$ represents the squared-magnitude of vectors inside the unit ball in the $y$-$z$ plane. So,

$$\|P_y \pm P_z\|^2 = \sup_{\|x\|\leq 1}(\|P_y(x) \pm P_z(x)\|^2) = \sup_{\|x\|\leq 1}(\|\langle x, y\rangle\|^2 + \|\langle x, z\rangle\|^2) = 1$$

So, $\|P_y + P_z\|^2 + \|P_y - P_z\|^2 = 1 + 1 = 2 \neq 4$. Therefore,

$$\|P_y + P_z\|^2 + \|P_y - P_z\|^2 \neq 2(\|P_y\|^2 + \|P_z\|^2)$$

So, the Parallelogram Law is broken. Thus, $CL(X) = CL(X, X)$ for $dim(X) > 1$ is not a Hilbert space. □

Now that we have some idea of what Banach and Hilbert spaces look like, we will extend our discussion on continuous maps between Banach and Hilbert spaces to a discussion of continuous maps from Banach and Hilbert spaces to the field $\mathbb{K}$.

# 2 Dual Spaces

## 2.1 Continuous Dual Space

**Definition.** *Let $CL(X, Y)$ be the set of continuous linear maps from a TVS $X$ to TVS $Y$. For a TVS $X$ over the field $\mathbb{K}$ (either real or complex), the continuous dual space (or simply dual space) $X^\star$ is $CL(X, \mathbb{K})$, the set of all continuous linear functionals $f : X \to \mathbb{K}$. Here, functionals are just maps from a TVS to the field.*

As we will see in Section 4, functionals in the dual spaces of Hilbert space represent states in quantum systems. So, understanding the properties of dual spaces will help us understand the nature of quantum states.

**Proposition 2.1.** *The dual space $B^\star$ of a Banach space $B$ over a field $\mathbb{K} = \mathbb{R}_{std}$ or $\mathbb{C}_{std}$ is also a Banach space with the operator norm defined by $\|f\| = \sup_{\|x\|\leq 1}(|f(x)|)$ for $f \in B^\star$.*

*Proof.* We first note that $\mathbb{K} = \mathbb{R}_{\text{std}}$ or $\mathbb{C}_{\text{std}}$ is Banach space (proof at [1]). So, by Proposition 1.3, $B^\star = CL(B, \mathbb{K})$ is a Banach space. □

An interesting question we may ask if what the dual spaces of dual spaces look like. There is a natural embedding (continuous injective map) $x \mapsto \phi_x$, from $X$ to $X^{\star\star} = (X^\star)^\star$ defined by $\phi_x : X^\star \to K$, $\phi_x(f) = f(x)$. This map characterizes the relationship between spaces and their double-duals.

**Definition.** *A TVS $X$ is reflexive if this natural embedding $x \mapsto \phi_x$ from $X$ to $X^{\star\star}$ is an isomorphism. Then, $X \cong X^{\star\star}$ via $x \mapsto \phi_x$. We may abuse notation and write $X = X^{\star\star}$.*

**Remark.** We note that since $x \mapsto \phi_x$ is a continuous injection, we only need to check that it is an isometric surjection to show that a TVS $X$ is reflexive.

We may naively think of the dual operation $^\star$ as a kind of complex conjugation and so just like the conjugate of a conjugate is the original constant, the dual of a dual is the original space. However, these next few examples will us that not all Banach spaces are reflexive. So, reflexivity of topological vector spaces, is in fact an interesting nontrivial concept.

**Proposition 2.2.** *$c_0$ is the space of sequences in $\ell^\infty$ that converge to 0. The dual space of $c_0$ is $\ell^1$.*

*Proof.* The idea behind this proof is to construct a continuous linear $f : \ell^1 \to c_0^\star$ by constructing a functional $f_t$ on $c_0$ for each element $t \in \ell^1$. show that $c_0 \subseteq \ell^1$ by defining a continuous functional defined by elements of $\ell^1$ from $c_0$ to $\mathbb{K}$. In other words, we need to construct an isomorphism $t \mapsto f_t$ from $\ell^1$ to $c_0^\star$. If this map is a linear, bijective, isometry, then it is an isomorphism.

Given $t \in \ell^1$, define the functional

$$f_t : c_0 \to \mathbb{K} \text{ by } f_t(x = \{x_i\}) = \sum t_i x_i$$

*Linearity:* Let $a, b \in \mathbb{K}$, $x, y \in c_0$ be arbitrary. Then,

$$f_t(ax + by) = \sum t_i(ax_i + bx_i) = a \sum t_i x_i + b \sum t_i y_i = af_t(x) + bf_t(y)$$

So, $f_t$ is linear. Additionally,

*Isometry:* To show that $f_t$ is an isometry, we first show that $f_t$ is bounded. Observe that

$$|f_t(x)| = |\sum t_i x_i| \leq \sum |t_i x_i| \leq \|x\|_\infty \sum |t_i| = \|x\|_\infty \|t\|_1$$

So, $f_t$ is also bounded by $\|t\|_1$. Now we show that the supremum of $|f_t(x)|$ is $\|t\|_1$ by finding a value $x \in c_0$ such that $|f_t(x)|$ is arbitrarily close to $\|t\|_1$. Let $\epsilon > 0$. Since $t \in \ell^1$, $\sum |t_i|$ converges. So, we may find some $N > 0$ such that for all $n > N$, $\sum_{i=n}^\infty |t_i| < \epsilon$. Then, let $x = \{x_i\}$ with $x_i = \overline{t_i}$ for $i \leq N$ and $x_i = 0$ for $i > N$. So, $x \in c_0$. Then,

$$|f_t(x) - \|t\|_1| = |\sum_{i=0}^{N} t_i x_i - \sum |t_i|| = |\sum_{i=N+1}^{\infty} |t_i|| < \epsilon$$

So, $\|f_t\| = \|t\|_1$. Thus, $f_t$ is an isometry.

*Injectivity:* This amounts to to showing that $f_t(x) = 0$ for all $x \in c_0$ if and only if $t = 0 \in \ell^1$ is the 0-sequence. If $t = 0 \in \ell^1$, then $f_t(x) = 0$ for all $x \in c_0$ obviously. If $f_t(x) = 0$ for all $x \in c_0$, then

$$f_t(e_i) = \sum t_j (e_i)_j = t_i = 0$$

Since $i$ is arbitrary, $t_i = 0$ for all $i$. So, $t = 0 \in \ell^1$.

*Surjectivity:* We now show that every continuous functional $f : c_o \to \mathbb{K}$ takes that form. Let $e_i$ be the basis sequence vector that has a 1 in the ith position and 0s everywhere else. Then, for an arbitrary continuous functional $f$,

$$|f(x)| = |\sum f(x_i e_i)| = |\sum x_i f(e_i)| \leq \sum |x_i||f(e_i)| \leq \|x\|_\infty \sum |f(e_i)|$$

Since $f$ is continuous, it must be bounded. So, $\sum |f(e_i)| < \infty$ and $\{f(e_i)\} \subseteq \ell^1$. Thus, $f = f_{\{f(e_i)\}}$.

Since the map $t \mapsto f_t$ from $\ell^1$ to $c_0^\star$ is a linear bijective isometry, it is an isomorphism. Thus, $c_0^\star \cong \ell^1$, or more simply $c_0^\star = \ell^1$. ☐

**Proposition 2.3.** $(\ell^1)^\star = \ell^\infty$

*Proof.* This proof is very similar (almost identical) to the proof in Example 2.2. We use functionals $f_t : \ell^1 \to \mathbb{K}$ for $t \in \ell^\infty$ of similar form, i.e. $f_t(x) = \sum t_i x_i$ to show $(\ell^1)^\star = \ell^\infty$. ☐

**Theorem 2.1.** *Not all Banach spaces are reflexive.*

*Proof.* We note that from Propositions 2.2, 2.3 that

$$(c_0)^\star = \ell^1 \text{ and } (c_0)^{\star\star} = (\ell^1)^\star = \ell^\infty$$

We know $\ell^\infty \neq c_0$ because $c_0$ is the set of sequences in $\ell^\infty$ that converge to 0, i.e. $\ell^\infty$ is a much larger space than $c_0$. Since $(c_0)^{\star\star} = \ell^\infty \neq c_0$, not all Banach spaces are reflexive. ☐

There is a subtlety to Theorem 2.1: there are models of set theory where Banach spaces that are not reflexive in one model, are reflexive in another model. For example, $\ell^1$ is not reflexive if we assume the axiom of choice (AOC), but is reflexive in models of set theory without the AOC. The non-reflexivity of $\ell^1$ with the Axiom of Choice relies on an important statement known as the Hahn-Banach Theorem [2].

**Theorem 2.2** (Hahn-Banach)**.** *For $X$ a normed linear space and $M \subseteq X$ a linear subspace, let $f \in M^\star$ be a bounded linear functional on M. Then, there exists a bounded linear functional $f' \in X^\star$ that extends $f$, i.e. $f'|_M = f$, and satisfies $\|f'\|_{X^\star} = \|f\|_{M^\star}$.*

**Proof Idea.** Although we will not prove the Hahn-Banach Theorem here, we may understand the idea behind it and its use of the axiom of choice. The Hahn-Banach Theorem relies on the axiom of choice in Zermelo-Fraenkel Set Theory (ZFC) to show that a partially ordered set, in which every totally ordered set has an upper bound, itself has at least one maximal element. This allows subsets and functionals defined on them to be ordered in such a way that sets are "greater" than their subsets and extended functions are "greater" than the functions they extend on those subsets. Eventually, you get a largest set $X$ and functional $f'$ on $X$.

The proof of the Hahn-Banach Theorem can be found here [2].

**Corollary 2.2.1.** *A corollary of the Hahn-Banach Theorem is that for any nonempty nontrivial normed vector space $X \neq \{0\}$, the dual space $X^\star$ is also nontrivial nonempty, i.e. $X^\star \neq \{0\}$.*

*Proof.* [3] Suppose $X \neq \{0\}$. Choose a nonzero vector $x \in X$ with $\text{span}(v) = Y \subseteq X$ a subspace. We may choose a linear functional $f \in Y^\star$ such that $f(v) = 1$. By Hahn-Banach Theorem, we can extend this functional to a functional $f'$ on $X^\star$. Since $f'(a) = f(a) = 1$ is nonzero $f' \neq 0$. So, there is a nonzero functional in $X^\star$. Thus, $X^\star \neq \{0\}$. □

**Remark.** Without the Hahn-Banach Theorem, the corollary 2.1 above is not always true. For example, there are models of set theory where $(\ell^\infty/c_0)^\star = \{0\}$. These models then yield reflexivity of $\ell^1$.

Now that we have some understanding of functionals in the dual space and an idea of why dual space reflexivity is an interesting and nontrivial concept, let us narrow our focus to studying the dual spaces of Hilbert spaces. We will find that Hilbert spaces are reflexive, which makes working with them and their dual spaces really nice!

## 2.2 Hilbert Space Duals

The fact that the inner product is the more primitive notion when compared to the norm makes Hilbert spaces very friendly. It gives rise to a nice structure that allows us to draw a bijective correspondence between a Hilbert space and its dual, preserve norms between elements in the space and their corresponding functionals, and define inner products on Hilbert space duals. Let us begin with some preliminary definitions and propositions.

**Definition.** *The orthogonal complement of $U \subseteq H$, a subspace of a Hilbert space $H$, is $U^\perp = \{y \in H \mid \langle x, y \rangle = 0$ for all $x \in H\}$.*

**Proposition 2.4.** *Let $H$ be a Hilbert space and $\psi : H \to \mathbb{K}_{std}$ be a continuous functional, i.e. $\psi \in H^\star$. Then, $\ker(\psi)$ is closed in $H$.*

*Proof.* We know

$$\ker(\psi) = \{x \in H \mid \psi(x) = 0\} = \psi^{-1}(\{0\})$$

$\{0\}$ is closed in $\mathbb{K}_{\text{std}}$ and $\psi$ is continuous. Thus, $\ker(\psi) = \psi^{-1}(\{0\})$ is closed in $H$. □

**Proposition 2.5.** *Let $H$ be a Hilbert space with dual $H^\star$. Then, for all $y \in H$, $\|\langle \cdot, y \rangle\|_{H^\star} = \|y\|_H$, where $\langle \cdot, y \rangle : H \to \mathbb{K}$ is an inner product map.*

*Proof.* The Cauchy-Schwarz Inequality tells us that $\langle x, y \rangle \leq \|x\|\|y\|$ for all $x \in H$. So, $\langle \cdot, y \rangle$ is bounded - more specifically,

$$\|\langle \cdot, y \rangle\|_{H^\star} \leq \|y\|_H$$

If we take $x = y$, then $\|y\|_H^2 = \langle y, y \rangle \leq \|\langle \cdot, y \rangle\|_{H^\star}\|y\|_H$. Then,

$$\|y\|_H \leq \|\langle \cdot, y \rangle\|_{H^\star}$$

Thus, $\|\langle \cdot, y \rangle\|_{H^\star} = \|y\|_H$. □

With those definitions and propositions, let us prove the Riesz Representation Theorem. It is probably the most important theorem in this article because it is what makes Hilbert spaces so nice to work with.

**Theorem 2.3** (Riesz Representation Theorem)**.** *For all continuous functionals $\psi \in H^\star$, there is a unique representation $f_\psi \in H$ such that $\psi(x) = \langle x, f_\psi \rangle$ for all $x \in H$. We call $f_\psi$ a representation of $\psi$ in $H$.*

*Proof.* Given, $\psi \in H$, we examine $N_\psi = \ker(\psi)$, a closed linear subspace of $H$ (by Proposition 2.4). If $N_\psi = H$, then $\psi(x) = 0$ for all $x \in H$ and $f_\psi = 0$. If $N_\psi \neq H$, then the orthogonal complement $N_\psi^\perp$ is nonempty.

So, we may find an element $g \in N_\psi^\perp$ such that $\|g\|_H = 1$. Then, for all $x \in H$, we may define an element $u_\psi \in N_\psi$ by

$$u_\psi = \psi(x)g - \psi(g)x$$

$u_\psi \in N_\psi$ because $\psi(u_\psi) = \psi(x)\psi(g) - \psi(g)\psi(x) = 0$.

Since $u_\psi \perp g$,

$$\begin{aligned} 0 = \langle u_\psi, g \rangle &= \langle \psi(x)g - \psi(g)x, g \rangle \\ &= \psi(x)\|g\|_H^2 - \psi(g)\langle x, g \rangle \\ &= \psi(x) - \psi(g)\langle x, g \rangle \end{aligned}$$

Then, $\psi(x) = \psi(g)\langle x, g \rangle = \langle x, \psi(g)g \rangle$. Thus, $f_\psi = \psi(g)g \in H$ is a representation of $\psi \in H^\star$.

To show uniqueness, suppose $\psi(x) = \langle x, f_\psi \rangle = \langle x, h_\psi \rangle$ for all $x \in H$. Subtracting, $\langle x, f_\psi - h_\psi \rangle = 0$ for all $x \in H$. Take $x = f_\psi - h_\psi$. Then,

$$\langle x, f_\psi - h_\psi \rangle = \langle f_\psi - h_\psi, f_\psi - h_\psi \rangle = \|f_\psi - h_\psi\|^2 = 0$$

Thus, $f_\psi = h_\psi$. ☐

**Corollary 2.3.1.** $\|\psi\|_{H^\star} = \|f_\psi\|_H$

*Proof.* $\|\psi\|_{H^\star} = \|\langle \cdot, f_\psi \rangle\|_{H^\star} = \|f_\psi\|_H$ by Proposition 2.5. ☐

Another useful idea that follows from the Riesz Representation Theorem is defining inner products on dual spaces, as seen in this Corollary:

**Corollary 2.3.2.** *The dual $H^\star$ is a Hilbert space $H$ is itself a Hilbert space.*

*Proof.* We know that the dual of a Banach space is a Banach space. So, we just need to define an inner product consistent with the norm on $H^\star$ to show that it is a Hilbert space. By the Riesz Representation Theorem (Theorem 2.3), for every $\psi, \phi \in H^\star$, there exist unique $f_\psi, f_\phi \in H$ such that $\psi(x) = \langle x, f_\psi \rangle_H$ and $\phi(x) = \langle x, f_\phi \rangle_H$. So, we may define

$$\langle \psi, \phi \rangle_{H^\star} = \langle f_\phi, f_\psi \rangle_H$$

One may check that this is a valid inner product. To check that this inner product is consistent with the operator norm on $H^\star$, take $\phi = \psi \in H^\star$. Then, by Corollary 2.3.1,

$$\langle \psi, \psi \rangle_{H^\star} = \langle f_\psi, f_\psi \rangle_H = \|f_\psi\|_H^2 = \|\psi\|_{H^\star}^2$$

So, $\langle \psi, \psi \rangle_{H^\star} = \|\psi\|_{H^\star}^2$. Thus, the norm and inner product are consistent. ☐

The Riesz Representation Theorem, in conjunction with Corollaries 2.3.1 and 2.3.2, gives rise to the reflexivity of Hilbert spaces and allows us to represent quantum states as functionals in a Hilbert dual space (as we will see in Section 4).

**Theorem 2.4.** *Hilbert spaces are reflexive.*

*Proof.* We use the definition of reflexivity to show that an arbitrary Hilbert space $H$ is reflexive. Specifically, we need to show that the natural embedding $x \mapsto \phi_x$ from $H$ to $H^{\star\star}$ defined by $\phi_x(\psi) = \psi(x)$ is an isometric surjection.

*Surjectivity:* Given an arbitrary $\phi \in H^{\star\star}$, the Riesz Representation theorem tells us that there is a unique $g_\phi \in H^\star$ such that for all $\psi \in H^\star$,

$$\phi(\psi) = \langle \psi, g_\phi \rangle_{H^\star}$$

Applying the Riesz Representation Theorem again, there is a unique $f_{g_\phi} \in H$ and a unique $f_\psi \in H$ such that

$$g_\phi(x) = \langle x, f_{g_\phi} \rangle_H \text{ and } \psi(x) = \langle x, f_\psi \rangle_H$$

for all $x \in H$. So, $f_{g_\phi} \in H \mapsto \phi \in H^{\star\star}$. Using the inner product definition on dual spaces in Corollary 2.3.2),

$$\phi(\psi) = \langle \psi, g_\phi \rangle_{H^\star} = \langle f_{g_\phi}, f_\psi \rangle_H = \psi(f_{g_\phi})$$

Thus,

$$\phi = \phi_{f_{g_\phi}}$$

So, for all $\phi \in H^\star\star$, there is an element $x \in H$ such that $x \mapsto \phi$ and $\phi(\psi) = \psi(x)$. So, $x \mapsto \phi_x$ is surjective.

*Isometry:* Applying Corollary 2.3.1 twice implies that $x \mapsto \phi_x$ is an isometry.

So, $x \mapsto \phi_x$ is an isometric surjective embedding, i.e. an isomorphism. Thus, $H$ is reflexive. □

So far, we have studied the properties of normed spaces, examined Banach and Hilbert spaces, and understood why dual spaces - especially dual Hilbert spaces - are important. We may dive even deeper into these spaces and introduce new notions of topologies on them. This next section covers two types of topologies - the weak and weak$^\star$ topologies - on the dual spaces of normed topological vector spaces.

## 2.3 Weak and Weak$^\star$ Topologies

Note: This section is self-contained - it is an interesting construction of topologies on the dual space, but it is not integral to understanding the formulation of quantum systems in this paper. Nonetheless, it provides us with additional insight into what are called the weak and weak$^\star$ topologies of dual spaces. A reason we care about these topologies is that they make otherwise non-compact spaces in the norm topology, compact.

Suppose we want to define a topology on the dual space of $X$ that makes every functional in $X^\star$ continuous in the "weakest" possible way, i.e. our topology should only define the minimum number of open sets in $X^\star$ for which all functionals in $X^\star$ are continuous.

**Definition.** *Let $X$ be a set, $Y$ be a topological space, and $F$ be a nonempty family of mappings from $X$ to $Y$. Define the topology $\tau_F$ on $X$ to be the collection of all unions and finite intersections of sets $f^{-1}(V)$ for $f \in F$ and open sets $V \in Y$.*

The reason why this new quantity $\tau_F$ defines a topology on X is because all maps $f \in F$ are now continuous and all properties of a topology (like $\phi, Y \in \tau_F$, closure under finite intersections, closure under arbitrary unions) are satisfied. Note that $\tau_F$ is the smallest (or coarsest) possible topology for which all $f \in F$ are continuous.

**Definition.** *The weak topology on a normed TVS $X$ is $\tau_{X^\star}$, i.e. the coarsest topology on $X$ such that all functionals $f : X \to \mathbb{K}$ in $X^\star$ remain continuous on $X$.*

Now, we extend this concept of weak topologies to define a topology on the dual space $X^\star$ of a normed TVS $X$ such that it is the coarsest topology on $X^\star$ such that all maps in $X^{\star\star}$ remain continuous.

**Definition.** *The weak$^\star$ topology on the dual space $X^\star$ of a normed TVS is $\tau_{X^{\star\star}}$, i.e. the coarsest topology on $X^\star$ such that all maps $\phi_x : X^\star \to \mathbb{K}$ for $x \in X$, defined by $\phi_x(f) = f(x)$ remain continuous. Here, the map $\phi_x \in X^{\star\star}$ is the natural embedding from $X$ into $X^{\star\star}$.*

**Remark.** *The weak$^\star$ topology is coarser than the weak topology, which is itself coarser than the norm topology [4]. If $X$ is reflexive, then the weak and weak$^\star$ topologies on $X^\star$ coincide.*

An important consequence of weak$^\star$ topologies is the fact that certain non-compact spaces in the norm topology are compact in the weak$^\star$ topology. The Banach-Alaoglu Theorem is a great example of this idea.

**Theorem 2.5** (Banach-Alaoglu)**.** *Let $X$ be a normed TVS. Then, the closed unit ball in $X^\star$ is compact in the weak$^\star$ topology.*

*Proof.* Although we will not prove this theorem here, it relies on Tychonoff's Theorem and thinking about the weak$^\star$ topology as a product topology. You can find the proof for the Banach-Alaoglu Theorem at [5] and [6]. □

**Example 2.1.** The closed unit ball in $(\ell^1)^\star$ is compact in the weak$^\star$ topology. However, the closed unit ball in $(\ell^1)^\star$ is not compact in the norm topology of $\ell^\infty$.

*Proof.* By the Banach-Alaoglu Theorem, the closed unit ball in $(\ell^1)^\star$ is weak$^\star$compact. For the norm topology case, let $e_n$ be the sequence of all 0s and a 1 in the nth coordinate. Then, $\|e_n\|_\infty = 1$ and is in the closed unit ball of $\ell^\infty$. Each sequence $e_n$ converges to 0, but $\|e_n - e_m\| = 1$ for all $n \neq m$. So, $\{e_n\}$ has no convergent subsequence in the closed unit ball in $\ell^\infty$. Thus, the closed unit ball in $\ell^\infty$ is not sequentially compact, and so not compact. □

Now that we have some understanding of Hilbert and Banach spaces, their duals, and their topologies, we will venture into a new topic integral to the mathematical foundation of quantum systems: C* algberas.

# 3 C* Algebras

In this section, we will build up an understanding of C* algebras and self-adjoint operators on Hilbert spaces that will be very useful in our exploration of quantum systems in Section 4.

## 3.1 C* Algebra Basics

**Definition.** *A Banach algebra is an associative algebra over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ that is also a Banach space.*

**Definition.** *A C* algebra $\mathcal{C}$ (pronounced "C-star" algebra, not to be confused with the dual star $\star$ of a normed space) is a Banach algebra with a map $x \mapsto x^*$ that satisfies the following properties:*

1. *It is an involution, i.e. $x^{**} = (x^*)^* = x$*
2. *For all $x, y \in \mathcal{C}$, $(x + y)^* = x^* + y^*$ and $(xy)^* = y^* x^*$*
3. *For all $a \in \mathbb{K}$, $x \in \mathcal{C}$, $(ax)^* = \overline{a} x^*$*
4. *For all $x \in \mathcal{C}$, $\|x^* x\| = \|x\|\|x^*\|$*

**Definition.** *For $\mathcal{C}$, a C* algebra, Her$(\mathcal{C}) = \{x \in \mathcal{C} \mid x^* = x\}$. We say the elements of this set are hermitian.*

**Proposition 3.1.** *For $\mathcal{C}$, a C* algebra, Her$(\mathcal{C})$ is a closed $\mathbb{R}$-subspace of $\mathcal{C}$.*

*Proof.* Let $\{x_i\} \subseteq \text{Her}(\mathcal{C})$ be a sequence that converges to $x$. Then, $x_i^* \to x^*$. Since $x_i \in \text{Her}(\mathcal{C})$, $x_i^* = x_i$. So, $x^* = x$. So, $x \in \text{Her}(\mathcal{C})$. Thus, Her$(\mathcal{C})$ is closed.

Additionally, for an element $x \in \text{Her}(\mathcal{C})$, and $a \in \mathbb{K}$,

$$ax \in \text{Her}(\mathcal{C}) \implies (ax)^* = \overline{a} x^* = \overline{a} x$$

But, $(ax)^* = ax$ because $ax$ is hermitian. So, $\overline{a} = a$. Thus, $a \in \mathbb{R}$.

Therefore, Her$(\mathcal{C})$ is a closed $\mathbb{R}$-subspace of $\mathcal{C}$. □

**Remark.** A closed *-subalgebra of a C* algebra is also a C* algebra. We may call such a *-subalgebra a C*-subalgebra. So, Her$(\mathcal{C})$ is a real C*-subalgebra of $\mathcal{C}$.

## 3.2 Operators on Hilbert Spaces as C* Algebras

We will associate quantum systems with C* algebras of operators on a Hilbert space in Section 4. So, let us examine how continuous linear operators on Hilbert spaces form C* algebras.

**Definition.** *The adjoint of an operator $f$ in $CL(H) = CL(H, H)$ on a Hilbert space $H$ is defined to be the map $f^*$ such that $\langle f(x), y\rangle_H = \langle x, f^*(y)\rangle_H$ for all $x, y \in H$.*

The adjoint of a map is well-defined and exists by the Riesz Representation Theorem (Theorem 2.3). If we let $f \in CL(H)$, and fix $y \in H$, then $x \mapsto \langle f(x), y\rangle$ is a functional in $H^\star$ - call it $\psi$. Then, by the Riesz Representation Theorem (Theorem 2.3), there is a unique $f_\psi \in H$ such that

$$\psi(x) = \langle f(x), y\rangle = \langle x, f_\psi\rangle$$

We define $f^* : H \to H$ by $f^*(y) = f_\psi$. Then,

$$\langle f(x), y\rangle_H = \langle x, f^*(y)\rangle_H$$

**Example 3.1.** The set of continuous operators $CL(H) = \{f : H \to H \mid f \text{ is continuous and linear}\}$ on a Hilbert space $H$ is a C* algebra with $f^*$ denoting the adjoint of $f$.

*Proof.* We know that $CL(H)$ is a Banach space by Proposition 1.3 with continuous addition and multiplication - so it is a Banach algebra. We simply check each of the four properties of the map $f \mapsto f^*$ for $f \in CL(H)$.

1. Given $f \in CL(H)$, and $x, y \in H$,

$$\langle f(x), y\rangle = \langle x, f^*(y)\rangle = \overline{\langle f^*(y), x\rangle} = \overline{\langle y, f^{**}(x)\rangle} = \langle f^{**}(x), y\rangle$$

So, $f^{**} = f$.

2. Given $f, g \in CL(H)$, and $x, y \in H$,

$$\begin{aligned}\langle (f+g)(x), y\rangle &= \langle f(x), y\rangle + \langle g(x), y\rangle \\ &= \langle x, f^*(y)\rangle + \langle x, g^*(y)\rangle \\ &= \langle x, (f^* + g^*)(y)\rangle\end{aligned}$$

So, $(f+g)^* = f^* + g^*$.
Additionally,

$$\begin{aligned}\langle (fg)(x), y\rangle &= \langle f(g(x)), y\rangle = \langle g(x), f^*(y)\rangle \\ &= \langle x, g^*(f^*(y))\rangle = \langle x, (g^*f^*)(y)\rangle\end{aligned}$$

So, $(fg)^* = g^*f^*$.

3. Given $a \in \mathbb{K}$, $f \in CL(H)$,

$$\langle (af)(x), y\rangle = a\langle x, f^*(y)\rangle = \langle x, \overline{a}f^*(y)\rangle$$

So, $(af)^* = \overline{a}f^*$.

4. We first show that given $f \in CL(H)$, $\|f^*\| = \|f\|$. $\|f^*(x)\|^2 = \langle f^*(x), f^*(x)\rangle = \langle (ff^*)(x), x\rangle \leq \|(ff^*)(x)\|\|x\| \leq \|f\|\|f^*(x)\|\|x\|$ (using Cauchy-Schwarz inequality). So,

$$\|f^*(x)\| \leq \|f\|\|x\| \implies \|f^*\| \leq \|f\|$$

Then, adjoint once more to get

$$\|f^{**}\| = \|f\| \leq \|f^*\|$$

Thus, $\|f\| = \|f^*\|$. So,

$$\|f^*f\| \leq \|f^*\|\|f\| = \|f\|^2$$

By above, $\|f^*(x)\|^2 \leq \|(ff^*)(x)\|\|x\| \leq \|(ff^*)\|\|x\|^2$. So, $\|f^*\|^2 \leq \|ff^*\|$. Adjoint to get $\|f^{**}\|^2 = \|f\|^2 \leq \|f^*f^{**}\| = \|f^*f\|$. Thus,

$$\|f^*f\| = \|f\|^2 = \|f\|\|f\| = \|f\|\|f^*\|$$

Therefore, $CL(H)$ is a C* algebra. ∎

**Remark.** Note that $CL(H)$ is a Banach space, but not a Hilbert space due to Propositions 1.3 and 1.5.

A special class of operators on a Hilbert space $H$ is the set of self-adjoint operators. In fact, these are quantities that will represent physical observables in a quantum system.

**Definition.** *An operator $f$ on a Hilbert space $H$ is self-adjoint if $f^* = f$. Equivalently, $f \in$ Her$(CL(H))$, i.e. $f$ is hermitian.*

**Proposition 3.2.** *Her$(CL(H))$ for a Hilbert space $H$ is a real Banach space.*

*Proof.* By Proposition 3.1, Her$(CL(H))$ is a closed $\mathbb{R}$-subspace of $CL(H)$, which is Banach. So, by Proposition 1.1, Her$(CL(H))$ is a real Banach space. □

The fact the the set of hermitian or self-adjoint operators on a Hilbert space forms a real Banach space is great! If we want to talk about real measurable observables as self-adjoint operators on a Hilbert space, they better be elements of a real vector space.

**Remark.** Since Her$(CL(H))$ is a closed $\mathbb{R}$-subspace of $CL(H)$, it is also a real C*-subalgebra of $CL(H)$. This is a point to remember when we visit the Gelfand-Naimark Theorem in Section 4. It draws a correspondence between arbitrary C* algebras and C*-subalgebras of $CL(H)$, which will allow us to formulate quantum mechanics in two equivalent ways - one formulation with reference to operators on Hilbert spaces, and another with reference to elements of C* algebras. If this doesn't parse now, don't worry about it - we'll dive into it in Section 4.

Something we may want to do now is draw a correspondence between arbitrary C* algebras and continuous linear (or equivalently bounded linear) maps on Hilbert spaces. That would allow us to talk about C* algebras in the language of Hilbert space operators.

# 4 Quantum Systems

In quantum mechanics, we may describe physical systems as C* algebras with a unit element. We may apply our knowledge of Banach spaces, Hilbert spaces, dual spaces, and C* algebras gained so far to define states and measurable quantities (called observables) on these systems.

## 4.1 Dirac-von Neumann Formulation

**Definition.** *Let $\mathcal{C}$ be a unital C* algebra. A Quantum System (or Operator System) is a *-closed subspace $\mathcal{Q}$ of $\mathcal{C}$ that contains $1$.*

Given a quantum system $\mathcal{Q}$, we can write down a set of axioms, known as the Dirac-von Neumann axioms, that define physical quantities on $\mathcal{Q}$.

**Axioms 4.1** (Dirac-von Neumann or Operator Algebra Formulation)**.** Given a quantum system $\mathcal{Q} \subseteq \mathcal{C}$, a closed C*-subalgebra of C* algebra $\mathcal{C}$ over a field $\mathbb{K}$,

1. An observable $A$ on a quantum system $\mathcal{Q}$ is a hermitian element of $\mathcal{Q}$, i.e. $A^* = A$.

2. Quantum states:

- A state $\psi$ on a quantum system $\mathcal{Q}$ is a positive functional on $\mathcal{Q}$ such that $\psi(1) = 1$, i.e. $\psi : \mathcal{Q} \to \mathbb{K}$, $\psi(x^*x) \geq 0$, $\psi(0^*0) = \psi(0) = 0$ and $\psi(1) = 1$. Call the space of states $\mathcal{S}(\mathcal{Q})$.
- A state that satisfies $\|\psi\| = 1$ is called a pure state - call the space of pure states $\mathcal{PS}(\mathcal{Q})$.

3. For a state $\psi$ on $\mathcal{Q}$, the expectation value of an observable $A$ is $\psi(A)$.

Our task now is to make sure that these axioms yield a "good" formulation of Quantum Mechanics. One check is to make sure that expectation values of observables yield real values so that they are actually measurable. Then, we might want to draw a correspondence between this Dirac-von Neumann formulation of Quantum Mechanics and another called the Hilbert space formulation that is more commonly used by physicists.

**Proposition 4.1.** *Let $\mathcal{Q}$ be a quantum system on a field $\mathbb{K}$. Given a state $\psi : \mathcal{Q} \to \mathbb{K}$ and an observable $A \in Her(\mathcal{Q})$, $\psi(A)$ is real.*

*Proof.* We know from Proposition 3.1 that $\mathrm{Her}(\mathcal{Q})$ is a closed $\mathbb{R}$-subspace of $\mathcal{Q}$. So, $\mathrm{Her}(\mathcal{Q})$ is a real C*-subalgebra of $\mathcal{Q}$. So, $\psi|_{\mathrm{Her}(\mathcal{Q})} : \mathrm{Her}(\mathcal{Q}) \to \mathbb{R}$. Thus, $\psi(A) \in \mathbb{R}$. □

Now that we know that the Dirac-von Neumann axioms are consistent with real measurements of observables, we may turn toward expressing this formulation of Quantum Mechanics using Hilbert spaces - a more commonly used formulation. To do so, we will study the relationship between states on a C* algebra and vectors in a Hilbert space, and observables in a C* algebra and operators on a Hilbert space.

## 4.2 Hilbert Space Representations

### 4.2.1 Representation of States

We know from Example 3.1 that $CL(H)$ is a C* algebra and that $\mathrm{Her}(CL(H))$ is a C*-subalgebra of $CL(H)$. So, let us try to represent vectors in a Hilbert space as pure state functionals on the C* algebra $CL(H)$.

**Proposition 4.2** (Pure state representation in a Hilbert space)**.** *If $\phi \in H$ for some Hilbert space $H$, then $\psi_\phi : CL(H) \to \mathbb{K}$ defined by $\psi_\phi(A) = \langle A\phi, \phi\rangle_H$ is a pure state if and only if $\|\phi\|_H = 1$.*

*Proof.* $(\implies)$ : If $\psi_\phi$ is a pure state, then $\psi_\phi(1) = \langle \phi, \phi\rangle = \|\phi\|_H = 1$. Thus, $\|\phi\|_H = 1$.

$(\impliedby)$ : If $\|\phi\|_H = 1$, then $\psi_\phi$ is a pure state because:

1. $\psi_\phi(1) = \langle \phi, \phi\rangle = \|\phi\|_H = 1$
2. $\psi_\phi(0) = \langle 0, \phi\rangle = 0$
3. $\psi_\phi(T^*T) = \langle T^*T(\phi), \phi\rangle = \langle T(\phi), T(\phi)\rangle = \|T(\phi)\|_H^2 \geq 0$

□

So, for a unit vector $\phi$ in a Hilbert space $H$, there is a pure state representation $\psi_\phi \in \mathcal{PS}(\mathcal{Q})$ in the dual space $(CL(H))^\star$ such that $\psi_\phi(A) = \langle A\phi, \phi\rangle_H$.

**Remark.** Although every unit vector $\phi$ in a Hilbert space has a corresponding pure state of the form $\psi_\phi$ from above, not all pure states $f$ have a corresponding unit vector representation $\phi_f$ in the Hilbert space [7].

We may write this more formally using the state-space notation from the Dirac-von Neumann axioms 4.1. Let $\mathcal{U}(H)$ be the set of unit vectors in $H$. Define

$$\alpha : H \to \mathcal{S}(\mathcal{Q}) \text{ by } \alpha(\phi) = \psi_\phi$$

with $\psi_\phi$ as in Proposition 4.2, then we may say

$$\alpha(\mathcal{U}(H)) \subseteq \mathcal{PS}(CL(H)) \subseteq \mathcal{S}(CL(H)) \subseteq (CL(H))^\star$$

### 4.2.2 Hilbert Space Representation of Observables

Just as we can represent elements of a Hilbert space $H$ as states on the C* algebra (or quantum system) $CL(H)$, we can represent bounded/continuous linear operators on H as elements of a C* algebra (or quantum system) with the following definition and theorem:

**Definition.** *A continuous linear (or equivalently bounded linear) operator $\eta : \mathcal{C} \mapsto \mathcal{D}$ between C* algebras $\mathcal{C}$ and $\mathcal{D}$ is a *-homomorphism if:*

1. *For all $x, y \in \mathcal{C}$, $\eta(xy) = \eta(x)\eta(y)$*
2. *For all $x \in \mathcal{C}$, $\eta(x^*) = (\eta(x))^*$*

*If $\eta$ is a bijective *-homomorphism, then we say $\eta$ is a *-isomorphism.*

**Theorem 4.1** (Gelfand-Naimark)**.** *An arbitrary C* algebra $\mathcal{C}$ is isometrically *-isomorphic to a C* subalgebra of $CL(H)$ for some Hilbert space $H$.*

*Proof.* Although we will not prove this theorem here, the proof uses concepts of state representation as discussed in Proposition 4.2. You can get a sense of the proof structure at [8]. □

By the Gelfand-Naimark Theorem (Theorem 4.1), quantum systems may be equivalently described by C*-subalgebras of $CL(H)$, the set of bounded/continuous linear operators on some corresponding Hilbert space $H$.

**Proposition 4.3.** *By the Gelfand-Naimark Theorem, let $\mathcal{Q}$ be a quantum system isometrically *-isomorphic to $\mathcal{O}$, a C*-subalgebra of $CL(H)$ for some Hilbert space $H$. Call this *-isomorphism $\kappa : \mathcal{Q} \to \mathcal{O}$. Then, for all observables $A \in \mathcal{Q}$, $\kappa(A) \in \mathcal{O} \subseteq CL(H)$ is self-adjoint.*

*Proof.* Let $A$ be an observable in $\mathcal{Q}$. Then, $A^* = A$. Since $\kappa$ is a *-isomorphism,

$$\kappa(A) = \kappa(A^*) = (\kappa(A))^*$$

So, $\kappa(A)$ is self-adjoint in $\mathcal{O} \subseteq CL(H)$. □

So, we may talk about the observables of a quantum system as bounded linear (or continuous linear) operators on a Hilbert space.

**Remark.** *Since we can represent all observables (hermitian elements) of a quantum system C* algebra* $\mathcal{Q}$ *by a self-adjoint element in* $\mathcal{O} \subseteq CL(H)$ *for some Hilbert space* $H$*,*

$$\kappa(Her(\mathcal{Q})) \subseteq Her(\mathcal{O}) \subseteq Her(CL(H)) \subseteq CL(H)$$

*However, it is not necessarily true that all self-adjoint elements in* $CL(H)$ *have a hermitian representation in* $\mathcal{Q}$*.*

## 4.3 Hilbert Space Formulation

Now, we combine all our results so far from Proposition 4.2 and 4.3 to define a set of Axioms similar to the Dirac-von Neumann Axioms in the language of Hilbert spaces instead of C* algebras.

**Axioms 4.2** (Hilbert Space Formulation)**.** *Let* $\mathcal{Q}$ *be a quantum system isometrically *-isomorphic to* $\mathcal{O} \subseteq CL(H)$*, a C*-subalgebra of* $CL(H)$ *for some Hilbert space* $H$*. Then,*

1. *An observable* $T \in \mathcal{O}$ *is a self-adjoint operator on* $H$*.*
2. *A state in our quantum system is given by a unit vector* $\phi$*, up to scalar multiples, in the Hilbert space* $H$*.*
3. *For a state* $\phi \in H$*, the expectation value of an observable* $T$ *is* $\langle T\phi, \phi\rangle = \langle \phi, T\phi\rangle$*.*

It is important to recall that

- For the set of unit vectors $\mathcal{U}(H)$ on a Hilbert space $H$, $\mathcal{U}(H) \hookrightarrow \mathcal{PS}(CL(H))$
- For a quantum system C*algebra $\mathcal{Q}$, Her($\mathcal{Q}$) $\hookrightarrow$ Her($CL(H)$) for some Hilbert space $H$

So, although they are not exactly the same formulation, one may move between the Gelfand-Naimark formulation of quantum systems and the Hilbert Space formulation using what's called the Gelfand-Naimark-Segal Construction [9].

## 4.4 Some Physics

Having constructed both the Dirac von-Neumann and Hilbert space formulations of Quantum systems, let us talk a little bit of quantum mechanics.

### 4.4.1 Eigenstates and Eigenvalues

**Definition.** *Working with the Hilbert space formulation over a field* $\mathbb{K}$*, let* $T \in CL(H)$ *be an observable and* $\phi \in H$ *be a state. Then, we say* $\phi$ *is an eigenstate of* $T$ *with eigenvalue* $\lambda \in \mathbb{K}$ *if* $T(\phi) = \lambda\phi$*.*

In quantum mechanics, states that are eigenstates of observable operators in the Hilbert space formulation are of significant importance. This is because the state of a system collapses upon measurement to an eigenstate of the observable being measured.

For example, we would like to know the position of a particle in a box. Before we measure its position, our lack of prior information on its whereabouts allow us to define its state as a combination of all possible positions in the box - it could be anywhere. Upon measurement, we collapse this state down to a single point in the box - this is the particle's "eigenstate of position". We call it an eigenstate of position because this collapsed state of the particle has an eigenvalue of its position associated with it. Mathematically, for some Hilbert space $H$, we would write this as

$$\hat{x}(\phi_x) = x\phi_x$$

where $\hat{x} \in CL(H)$ is the position measurement operator (an observable), $\phi_x \in H$ is the eigenstate of position (a unit vector in $H$) associated with the position $x$ in space, the eigenvalue of $\phi_x$ upon position measurement.

**Remark.** In quantum mechanics, operators on a Hilbert space are sometimes denoted by a "hat" ˆ over them because the letter associated with them represents their eigenvalues that are denoted by the same letter.

Since measurements of real objects should yield real values, we should check that the eigenvalues associated with observables are real.

**Proposition 4.4.** *Let $T$ be an observable on some Hilbert space $H$ and $\phi \in H$ be an eigenstate of $T$ with eigenvalue $\lambda$. Then, $\lambda \in \mathbb{R}$.*

*Proof.* Since $T$ is an observable, $T^* = T$. Since $\phi$ is a state, $\|\phi\|^2 = \langle \phi, \phi \rangle = 1$. So, by axiom 3 of the Hilbert space formulation,

$$\lambda = \lambda\langle \phi, \phi \rangle = \langle T(\phi), \phi \rangle = \langle \phi, T(\phi) \rangle = \lambda^* \langle \phi, \phi \rangle = \lambda^*$$

Thus, $\lambda = \lambda^* \implies \lambda \in \mathbb{R}$. □

So, we know by Propositions 4.1 and 4.4 that the expectation values of observables and the actual measurements of observables will yield real values - great! But, what exactly do we mean by "measurement"? It's a word used a lot in this section, but never really discussed mathematically.

### 4.4.2 Measurements

We said a quantum system with some original state $\phi$ may collapse onto an eigenstate $\phi'$ after a measurement is made. So, before measurement, there is a probability that the system will be found in state $\phi'$ upon measurement. So, the state before measurement and the probabilities of collapse associated with each possible state of the system fully characterize what a measurement is.

**Axioms 4.3** (The Born Rule)**.** *Working with the Hilbert space formulation, for a quantum system in state $\phi \in H$, the probability of finding the system in a state $\phi'$ upon measurement is given by $|\langle \phi', \phi \rangle|^2$.*

We may translate the Born Rule axiom above into the Dirac-von Neumann formulation as such: Define $P_{\phi'} : H \to H$ by $P_{\phi'}(\phi) = \langle \phi, \phi' \rangle \phi'$. $P_{\phi'}$ is the projection operator - it projects states on the $\phi'$ vector in the Hilbert space $H$. Then,

$$|\langle \phi', \phi \rangle|^2 = \langle \phi, \phi' \rangle \langle \phi', \phi \rangle = \langle P_{\phi'}(\phi), \phi \rangle$$

Let $\psi_\phi \in \mathcal{PS}(CL(H))$ be the Dirac-von Neumann state representation of $\phi$ given by Proposition 4.2 and $\kappa : \mathcal{Q} \to \mathcal{O}$ be the *isomorphism from the associated quantum system $\mathcal{Q}$ to the C*-subalgebra $\mathcal{O}$ of $CL(H)$. Then,

$$|\langle \phi', \phi \rangle|^2 = \langle P_{\phi'}(\phi), \phi \rangle = \psi_\phi \left( \kappa^{-1}(P_{\phi'}) \right)$$

This is true provided $P_{\phi'}$ has a representation in $\mathcal{Q}$, the associated quantum system C* algebra. The probability associated with finding the system in a state $\phi'$ upon measurement is just the expectation value of the C* algebra representation of the projection operator onto $\phi'$.

Therefore, we may think of measurements as projections of original states onto other possible states of the system.

#### 4.4.3 Commentary on the use of both formulations

Both the Dirac-von Neumann and the Hilbert Space formulations of quantum systems are important to quantum physics.

The Hilbert space formulation is more commonly used and taught in simple quantum systems where quantities like particle number don't change, references frames are inertial, etc. It also provides an easier understanding of quantum systems because one can use the niceties of linear algebra to talk about physical measurements on quantum states as inner products of linear operators acting on state vectors in a Hilbert space. This is as opposed to describing physical measurements as functionals acting on elements of a C* algebra in the Dirac-von Neumann formulation.

However, the Dirac-von Neumann formulation is very useful when analyzing more complicated quantum systems, where quantities like particle number and reference frame accelerations change. For example, (in Minkowski spacetime) a quantum system in its lowest energy (ground) state with no particles as seen by an inertial observer may be seen to have a "thermal bath" of infinitely many particles by an accelerating observer. It is difficult to describe this system and the disconnect between observers' measurements using the Hilbert space formulation of quantum mechanics because the operators and state spaces themselves are different for each observer.

However, there is a C* algebra that is isometrically *-isomorphic to the *subalgebra of bounded linear operators on both observers' Hilbert spaces by the Gelfand-Naimark Theorem. So, this C* algebra captures both observers' Hilbert space representations of the system. So, it is advantageous to use the Dirac-von Neumann formulation of quantum mechanics when talking about complicated variable (particle number/mass/acceleration dependent) systems - this is the case in Quantum Field Theory and General Relativity [10].

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