n-Qubit Teleportation with a 2n-Qubit Binary Channel

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Quantum teleportation has been an important and exciting scientific advancement in the field of quantum simulation and computing since the 1990s. In this paper, we first examine, implement, and test an existing protocol for teleporting 2 arbitrary qubits using a teleportation channel-state of 4 intermediary entangled qubits. We then extend this protocol to devise, implement, test, and analyze the teleportation of an arbitrary number of n qubits using a quantum teleportation channel-state of $2n$ intermediary entangled qubits.

I. INTRODUCTION

Quantum teleportation is a phenomenon that describes the transfer of information about a quantum state between individuals using channels of entangled states. The importance of quantum teleportation can be seen in quantum cryptography, quantum communication, and quantum computing algorithms.

Since the 1990s, researchers have been developing protocols for teleporting qubits using a variety of quantum teleportation channels that include EPR pairs, 3-qubit GHZ states, 3-qubit W-states, cluster states, and more. However, these protocols and channel-states differ depending upon the number of qubits being teleported.

This poses an immediate question: How can we develop a protocol for teleporting an arbitrary number of qubits using a single teleportation channel of entangled qubits? Finding such a protocol would allow us to define a general process for quantum teleportation regardless of the number of qubits being teleported.

To answer this question, we first examine and test work done by Rigolin [1] on the teleportation of an arbitrary 2-qubit state using a 4-qubit teleportation channel. We then extend this to develop and test a protocol to deterministically teleport an arbitrary n-qubit state using a $2n$-qubit teleportation channel. We then test this protocol on a quantum circuit to teleport a 3-qubit initial state, and discuss the useful structure of this circuit for building circuits to teleport higher numbers of qubits.

It is important to note that all quantum states in this paper are written in Qiskit notation, i.e. the 0th qubit is the rightmost qubit in the ket ($|q\rangle = |q_{n-1}...q_1q_0\rangle$).

II. TWO-QUBIT TELEPORTATION PROTOCOL

In this section, we implement and test Rigolin's protocol for teleporting an arbitrary 2-qubit state $|\text{Alice}\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ from Alice to Bob.

A. Methodology

The channel-state used for this protocol is

$$\begin{aligned} |g\rangle &= \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\ &= (\text{SWAP}_{0,3})(|\Phi^+\rangle \otimes |\Phi^+\rangle) \end{aligned} \tag{1}$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a bell state. So, the total initial state is

$$|\psi_{\text{initial}}\rangle = |g\rangle \otimes |\text{Alice}\rangle$$

In practice, we would now give the 4th and 5th qubits to Bob, and let Alice keep the first four qubits for implementing the sender-side teleportation protocol. The send side protocol involves entangling Alice's two initial qubits with her two channel-state qubits by performing a CNOT operation on qubit 0 and qubit 2, followed by a Hadamard operation on qubit 0. She repeats these two steps on qubit 1 and qubit 3 respectively. The final state before any measurements is:

$$\begin{aligned} |\psi_{\text{final}}\rangle &= (\text{H}_1)(\text{CNOT}_{1,3})(\text{H}_0)(\text{CNOT}_{0,2})|\psi_{\text{initial}}\rangle \\ &\propto (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)|0000\rangle \\ &+ (a|00\rangle - b|01\rangle + c|10\rangle - d|11\rangle)|0001\rangle \\ &+ (a|00\rangle + b|01\rangle - c|10\rangle - d|11\rangle)|0010\rangle \\ &+ (a|00\rangle - b|01\rangle - c|10\rangle + d|11\rangle)|0011\rangle \\ &+ (b|00\rangle + a|01\rangle + d|10\rangle + c|11\rangle)|0100\rangle \\ &+ (-b|00\rangle + a|01\rangle - d|10\rangle + c|11\rangle)|0101\rangle \\ &+ (b|00\rangle + a|01\rangle - d|10\rangle - c|11\rangle)|0110\rangle \\ &+ (-b|00\rangle + a|01\rangle + d|10\rangle - c|11\rangle)|0111\rangle \\ &+ (c|00\rangle + d|01\rangle + a|10\rangle + b|11\rangle)|1000\rangle \\ &+ (c|00\rangle - d|01\rangle + a|10\rangle - b|11\rangle)|1001\rangle \\ &+ (-c|00\rangle - d|01\rangle + a|10\rangle + b|11\rangle)|1010\rangle \\ &+ (-c|00\rangle + d|01\rangle + a|10\rangle - b|11\rangle)|1011\rangle \\ &+ (d|00\rangle + c|01\rangle + b|10\rangle + a|11\rangle)|1100\rangle \\ &+ (-d|00\rangle + c|01\rangle - b|10\rangle + a|11\rangle)|1101\rangle \\ &+ (-d|00\rangle - c|01\rangle + b|10\rangle + a|11\rangle)|1110\rangle \\ &+ (d|00\rangle - c|01\rangle - b|10\rangle + a|11\rangle)|1111\rangle \end{aligned} \tag{2}$$

Once $|\psi_{\text{final}}\rangle$ is obtained, Alice can make measurements on her four qubits, and relay her measurements to Bob.

Then, Bob applies gate operations on his two qubits corresponding to Alice's measurements, as detailed in TABLE I.

TABLE I. The first column is a possible measurement that Alice might make on the final state $|\psi_{\text{final}}\rangle$ of her four qubits $|q_3q_2q_1q_0\rangle$. The second column provides the gate operations Bob must apply to his two qubits $|q_5q_4\rangle$ dependent upon the corresponding measurement.

Alice's Measurement	Bob's Gate Operations
$\|0000\rangle$	I
$\|0001\rangle$	σ_4^z
$\|0010\rangle$	σ_5^z
$\|0011\rangle$	$\sigma_5^z\sigma_4^z$
$\|0100\rangle$	σ_4^x
$\|0101\rangle$	$\sigma_4^z\sigma_4^x$
$\|0110\rangle$	$\sigma_5^z\sigma_4^x$
$\|0111\rangle$	$\sigma_5^z\sigma_4^z\sigma_4^x$
$\|1000\rangle$	σ_5^x
$\|1001\rangle$	$\sigma_4^z\sigma_5^x$
$\|1010\rangle$	$\sigma_5^z\sigma_5^x$
$\|1011\rangle$	$\sigma_5^z\sigma_4^z\sigma_5^x$
$\|1100\rangle$	$\sigma_5^x\sigma_4^x$
$\|1101\rangle$	$\sigma_4^z\sigma_5^x\sigma_4^x$
$\|1110\rangle$	$\sigma_5^z\sigma_5^x\sigma_4^x$
$\|1111\rangle$	$\sigma_5^z\sigma_4^z\sigma_5^x\sigma_4^x$

We now implement this protocol for teleporting two arbitrary qubits using a 4-qubit channel-state using a Quantum Circuit, as shown in FIG. 1.

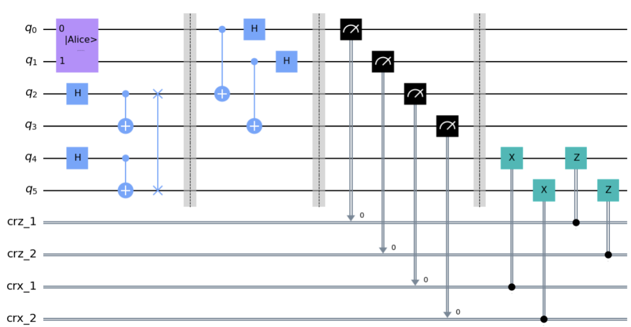

FIG. 1. Circuit for teleporting an arbitrary 2-qubit state from Alice to Bob using the $|g\rangle$ teleportation channel-state. The 1st (leftmost) section of this circuit initializes Alice's state and creates the channel-state; the 2nd section entangles Alice's initial qubits with 2 of the channel-state qubits; the 3rd section measures Alice's 4 total qubits and sends the information to Bob; the 4th section runs the gate operations on Bob's two qubits corresponding to Alice's measurement.

B. Results and Discussion

We run the 2-qubit teleportation circuit on IBM's `qasm-simulator` and their physical hardware with an initial state of $|\text{Alice}\rangle = \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{4}|01\rangle + \frac{3}{4}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$. Although IBM's quantum hardware does not yet support conditional gates in quantum circuits [2], we work around this by implementing deferred measurements using [3] control gates to replicate the behavior of applying σ_x and σ_z gates on Bob's qubits after Alice's measurements.

FIG. 2. shows the probability of measuring Bob's final state after teleportation in each of the $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ on both the simulator and IBM's quantum hardware computer `ibmq-jakarta` using 8192 shots (number of times the circuit is run).

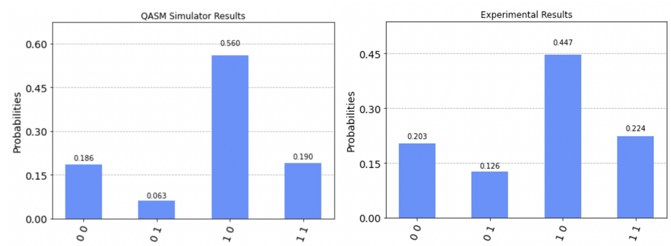

FIG. 2. Initial State: $|\text{Alice}\rangle = \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{4}|01\rangle + \frac{3}{4}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$. Probabilities of measuring Bob's final state after his gate operations in each of the $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ states after running the circuit on the `qasm-simulator` (left) and on the `ibmq-jakarta` quantum computer (right) with 8192 shots.

We calculate the error of each result by running Bob's final state through a disentangler that inverts the gates that created Alice's original state. Ultimately, the state would theoretically be $|00\rangle$ after disentanglement. The error in measurement is just the total probability of finding the qubits outside the $|00\rangle$ state. Thus,

$$\text{Error rate} = \left[\text{Prob}(|01\rangle) + \text{Prob}(|10\rangle) + \text{Prob}(|11\rangle)\right] 100\%$$

Using this error protocol, we measure the error rate of running the circuit on the `qasm-simulator` to be 0.002% and the error rate of running the circuit on the quantum computer `ibmq-jakarta` to be 1.942%. The reason for a larger error rate on the quantum hardware due to lower relaxation/coherence times on `ibmq-jakarta` than on the `qasm-simulator`.

The results in FIG. 2. demonstrate the effectiveness of implementing our teleportation circuit and protocol for teleporting two qubits. Using quantum hardware with lower decoherence times will help reduce the error in the physical implementation of our protocol.

III. GENERAL N-QUBIT TELEPORTATION PROTOCOL

After developing a protocol for teleporting two qubits, a natural next step is to develop a protocol for teleporting an arbitrary state of n qubits from Alice to Bob. We may write this state as

$$|\text{Alice}\rangle = \sum_{i=0}^{2^n-1} c_i |\text{bin}_n(i)\rangle$$

where $c_i \in \mathbb{C}$ and $\text{bin}_n(i)$ is the binary representation of i using n digits.

A. Methodology

1. *The Channel-State*

To find the channel-state for teleporting n qubits, let us first examine the 2-qubit teleportation channel-state: $|g\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) = \frac{1}{2}(|00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|11\rangle) = \frac{1}{2}\sum_{i=0}^{3} |\text{bin}_2(i)\rangle|\text{bin}_2(i)\rangle$.

So, we may infer a channel-state for the n-qubit teleportation protocol:

$$|g\rangle = \frac{1}{2^{n/2}} \sum_{i=0}^{2^n-1} |\text{bin}_n(i)\rangle|\text{bin}_n(i)\rangle$$

Thus, we would have a $2n$-qubit channel $|q_{3n-1}...q_n\rangle$ to teleport an arbitrary n-qubit state $|q_{n-1}...q_1q_0\rangle$.

Close inspection of the channel-state for low n values indicated the possibility of using SWAP gates on EPR pairs to generate the channel state. Using this approach as a base, we develop a classical algorithm that is able to verify whether operations performed on the n EPR pairs (that form the basis of the channel-state) lead to a channel-state that resembles $|g\rangle$. We have implemented this algorithm and successfully identified SWAP gate combinations that generate the channel-state for $n = 2, 3$ and 4. The corresponding circuit can be found in FIG. 3.

However, attempts to replicate this success for values of $n \geq 5$ were not successful, suggesting that further research into this matter may be needed. The runtime of the classical algorithm is $O(n!)$. The distinct advantage of the classical algorithm we have developed is that it can also be extended to test whether combinations other than just SWAP gates can form the channel-state.

2. *Alice's Gate Operations*

Once we have created our channel-state, the total initial state of the system is

$$|\psi_{\text{initial}}\rangle = |g\rangle \otimes |\text{Alice}\rangle$$

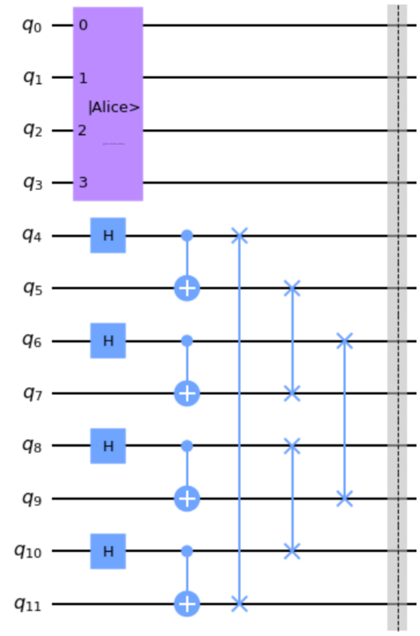

FIG. 3. Circuit for teleporting an arbitrary 4-qubit state from Alice to Bob using the $|g\rangle$ channel-state, generated by the classical algorithm

Now, Bob takes the latter half $|q_{3n-1}q_{3n-2}...q_{2n}\rangle$, and Alice keeps the former half $|q_{2n-1}q_{2n-2}...q_n\rangle$ along with her own state $|\text{Alice}\rangle = \sum_{i=0}^{2^n-1} c_i|\text{bin}_n(i)\rangle$ in $|q_{n-1}q_{n-2}...q_1q_0\rangle$. The structure of the latter half of the $|g\rangle$-state ensures that Bob receives all possible combinations needed to transform his state back to Alice's.

To understand the gate operations Alice must run on her qubits, we look to the 2-qubit circuit once again. We note that the final state after Alice's operations is $|\psi_{\text{final}}\rangle = (\text{H}_1)(\text{CNOT}_{1,3})(\text{H}_0)(\text{CNOT}_{0,2})|\psi_{\text{initial}}\rangle$. Thus, we may infer once more that the final state for the n-qubit teleportation case is

$$|\psi_{\text{final}}\rangle = \left[\prod_{i=0}^{n-1} (\text{H}_i)(\text{CNOT}_{i,i+n})\right] |\psi_{\text{initial}}\rangle$$

This visually corresponds to a staircase of Control NOT and Hadamard Gates that Alice sends her qubits through.

3. *Measurements and Bob's Gate Operations*

Once this final state has been achieved, Alice may measure her $2n$ qubits and send the information to Bob so that he may apply the requisite gate operations to his n qubits to retrieve Alice's original state.

Extending the information from TABLE I. to the n-qubit case, we note that if the ith qubit within the latter half of Alice's qubits $|q_{2n-1}...q_n\rangle$ is measured to be a 1, then Bob must apply the σ^x_{i+n} gate operation onto the $(i+n)$th qubit in his state $|q_{3n-1}...q_{2n}\rangle$; and then if the

ith qubit within the former half of Alice's state $|q_{n-1}...q_0\rangle$ is measured to be a 1, then Bob must apply a σ^z_{i+2n} gate operations onto the $(i+2n)$th qubit in his state.

Formally, if Alice's state after measurement is $|q_{2n-1}...q_j...q_n...q_i...q_0\rangle$, then Bob's total gate operations is defined by

$$\hat{\mathcal{K}} = \left[\prod_{i=0}^{n-1} \left(\sigma^z_{i+2n}\right)^{\delta_{q_i 1}}\right] \left[\prod_{j=n}^{2n-1} \left(\sigma^x_{j+n}\right)^{\delta_{q_j 1}}\right]$$

where δ_{ab} is the Kronecker delta on a and b. Then Bob can retrieve Alice's initial state by applying $\hat{\mathcal{K}}$ onto his state:

$$\hat{\mathcal{K}}|q_{3n-1}...q_{2n}\rangle = |\text{Alice}\rangle$$

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B. Results and Discussion

Let us now implement this circuit for teleporting an arbitrary state of 3 qubits. Using the methodology explained in the previous section, we may draw the circuit to teleport 3 qubits, as depicted in FIG. 4.

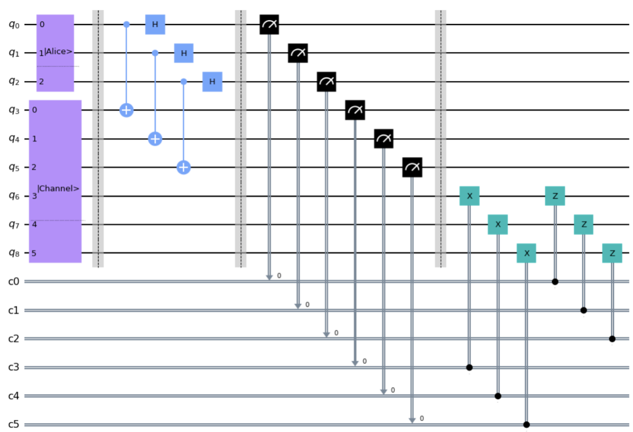

FIG. 4. Circuit for teleporting an arbitrary state of 3 qubits from Alice (qubits 0, 1, 2) to Bob (qubits 6, 7, 8)

The channel-state in this circuit is

$$|g\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle) + \frac{1}{\sqrt{8}}(|100\rangle + |101\rangle + |110\rangle + |111\rangle) \tag{3}$$

We now implement this circuit on the `qasm-simulator` with the initial state of

$$|\text{Alice}\rangle = \frac{1}{\sqrt{2}}(\frac{\sqrt{3}}{4}|000\rangle + \frac{1}{4}|001\rangle + \frac{3}{4}|010\rangle + \frac{\sqrt{3}}{4}|011\rangle)) + \frac{1}{\sqrt{2}}(\frac{\sqrt{3}}{4}|100\rangle + \frac{1}{4}|101\rangle + \frac{3}{4}|110\rangle + \frac{\sqrt{3}}{4}|111\rangle) \tag{4}$$

We run our teleportation circuit on the **`qasm-simulator`** with 1,000,000 shots and plot the histogram results, as shown in FIG. 5. We note that our protocol successfully teleports the 3-qubit state with an error rate of $9.69 \times 10^{-5}\%$.

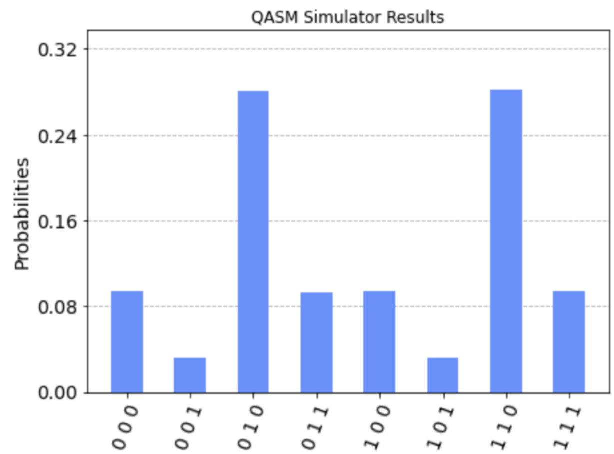

FIG. 5. Probabilities of measuring Bob's final state after his gate operations in each of the $|000\rangle$, $|001\rangle$, $|010\rangle$, $|011\rangle$, $|100\rangle$, $|101\rangle$, $|110\rangle$, $|111\rangle$ states after running the circuit on the **`qasm-simulator`** with 1,000,000 shots.

Although we ran out circuit on the **`qasm-simulator`**, we were unable to run it successfully on IBM's existing quantum hardware available to us. With hardware that can handle large qubit capacity and large relaxation/coherence times, we will be able to not only physically implement the 3-qubit teleportation protocol, but also an arbitrary n-qubit teleportation protocol. One may construct the circuit to teleport an arbitrary state of n qubits using the methodology outlined in the previous section.

IV. CONCLUSION

To summarize, in this paper, we have implemented and tested a teleportation protocol for teleporting arbitrary states of n-qubits by extending the base case of teleporting 2-qubits (as in Rigolin [1]). Although implementation of our work on physical quantum systems is limited by current day technological capabilities on quantum volume and state relaxation/coherence times, we hope our findings are beneficial for future uses of quantum teleportation as quantum technology continues to improve.

One of the most significant applications of quantum teleportation is long-range quantum communication networks [4] - also known as the 'quantum internet' [5] - teleportation of more than 2 qubits may support a robust quantum internet. Thus, active further research into teleportation of n qubits - especially the efficient generation and utilization of channel-states, and improvements in fidelity - would be beneficial. The usage of quantum teleportation in quantum repeaters to improve the feasibility of quantum communication is also being discussed [6]. Additionally, questions regarding conditional gates we encounter in our study of quantum teleportation have spurred research questions in the development of 'quantum gate teleportation'. Another significant application is quantum gate teleportation of spatially separated qubits in trapped-ion processors [7]. Furthermore, quantum teleportation from light beams to vibrational states of macroscopic diamond (fidelity achieved 90%, much higher than classical 2/3 limit) has major implications for optomechanical quantum control, and quantum information science [8].

V. FUTURE WORK

Further work that may be done for this project includes the identification of a generalized protocol for generating the $2n$-qubit channel-states using SWAP gates on EPR pairs, and the analysis of viability and gate-resource efficiency of such a protocol. Writing this protocol with low runtimes will be paramount to quick and easy generation of the channel-states, facilitating more effective and efficient long distance quantum communication.

Another area that warrants further examination is identifying more resource efficient channel-states for teleporting n-qubits. Although we have used a specific form of $2n$-qubit channel-states for our teleportation protocol, identifying other channel-states for easier creation capability and resource efficiency will be beneficial for physical implementations of quantum teleportation.

VI. ACKNOWLEDGMENTS

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