SUPERSYMMETRY AND THE WITTEN INDEX

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Abstract

In this exposition, we introduce the notion of supersymmetric quantum sysytems and the Witten index. The Witten index counts the difference between the number of bosonic and fermionic zero energy states in a supersymmetric quantum system. We apply the Atiyah–Singer index theorem for chiral Dirac operators to study the Witten index of an $\mathcal{N} = 1$ supersymmetric nonlinear sigma model.

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1 Supersymmetric Quantum Mechanics

1.1 Setup

Definition 1.1. A **quantum system** is a collection $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ of a Hilbert space $\mathcal{H}$ endowed with a hermitian inner product $\langle \cdot, \cdot \rangle$.

Various operators can act on the elements of the Hilbert space $\mathcal{H}$ of a quantum system. The **observable** operators are those that are hermitian. An operator of particular import is the **Hamiltonian** H, whose eigenvalues correspond to the quantized **energies** of the system.

A suspersymmetric quantum system is a quantum system with additional structure — in particular, the Hilbert space $\mathcal{H}$ will be graded.

Definition 1.2. A $_2$**-graded Hilbert space** or **super Hilbert space** is a collection $(\mathcal{H}, K)$ of a Hilbert space $\mathcal{H}$ and an operator $K : \mathcal{H} \to \mathcal{H}$ such that $K^2 = 1$. We call K the **grading operator**.

Since $K^2 = 1$, a super Hilbert space may be decomposed as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into the positive (**even-graded**) $\mathcal{H}_+$ and negative (**odd-graded**) $\mathcal{H}_-$ eigenspaces of K.

Definition 1.3. We say that an operator $A : \mathcal{H} \to \mathcal{H}$ is **even** with respect to K if it commutes with K and **odd** if it anticommutes with K. So, an even operator is a linear map $A_e : \mathcal{H}_\pm \to \mathcal{H}_\pm$ and an odd operator is a linear map $A_o : \mathcal{H}_\pm \to \mathcal{H}_\mp$.

Definition 1.4. An $\mathcal{N} = n$ **supersymmetric quantum system** is a collection $(\mathcal{H}, (-1)^F, H, Q_{i=1,\dots,n})$ of a Hilbert space $\mathcal{H}$, grading operator $(-1)^F$, and operators $H : \mathcal{H} \to \mathcal{H}$ and $Q_{i=1,\dots,n} : \mathcal{H} \to \mathcal{H}$ such that

- $(\mathcal{H}, (-1)^F)$ is a super Hilbert space. We write $\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$, where $\mathcal{H}^B$ is even-graded and $\mathcal{H}^F$ is odd graded. The states in H^B are called **bosonic** and the states in H^F are called **fermionic**.
- H is even-graded and hermitian. H is called the **Hamiltonian**.
- $Q_{i=1,\dots,n}$ are odd-graded and hermitian (typically called **supercharges**) such that

$$\{Q_i, Q_j\} = 2\delta_{ij} H$$

We note that since each Q_i is odd-graded, they obey $Q_i : \mathcal{H}^B \to \mathcal{H}^F$ and vice versa. So, we may think of the supercharges as coming from a chain complex of length 2

$$\mathcal{H}^B \xrightarrow{Q_i^B} \mathcal{H}^F$$

where Q_i^B is the restriction of Q_i to $\mathcal{H}^B$ and its adjoint $(Q_i^B)^* = Q_i^F$ is the restriction of Q_i to $\mathcal{H}^F$. The Euler characteristic of this complex is an important invariant called the **index** of Q_i, denoted $\mathrm{ind}(Q)$. More on this later — we will first examine the relationship between the grading operator, supercharges, and Hamiltonian more closely.

1.2 Operator Relationships

Let $(\mathcal{H}, (-1)^F, H, Q_{i=1,\dots,n})$ be an $\mathcal{N} = n$ supersymmetric quantum system.

Proposition 1.1. $[Q_i, H] = 0$ for all $i = 1, \dots, n$.

Proof. Since $\{Q_i, Q_j\} = 2\delta_{ij} H$,

$$H = \frac{1}{2} Q_i^2 = \frac{1}{2n} \sum_{j=1}^{n} Q_j^2$$

So, $[Q_i, H] = \frac{1}{2}[Q_i, Q_i^2] = 0$. □

Remark 1.1. Since each Q_i is hermitian and $H = \frac{1}{2}Q_i^2 = \frac{1}{2n}\sum_{j=1}^{n} Q_j^2$, the eigenvalues (energies) of H are nonnegative.

Proposition 1.2. $[(-1)^F, H] = 0$

Proof.

$$\begin{aligned}
[(-1)^F, H] &= \frac{1}{2}[(-1)^F, Q_i^2] \\
&= \frac{1}{2} Q_i [(-1)^F, Q_i] + \frac{1}{2}[(-1)^F, Q_i] Q_i \\
&= Q_i (-1)^F Q_i + (-1)^F Q_i^2 \text{ (since } Q_i \text{ is odd-graded)} \\
&= Q_i (-1)^F Q_i - Q_i (-1)^F Q_i \text{ (since } Q_i \text{ is odd-graded)} \\
&= 0
\end{aligned}$$

□

Let $\mathcal{H}_{(m)} \subseteq \mathcal{H}$ denote the eigenspace of the Hamiltonian H corresponding to the eigenvalue E_m, with $E_0 = 0$ and $M > m \implies E_M > E_m$. We may decompose the Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = \bigoplus_m \mathcal{H}_{(m)}$$

Since Q_i and $(-1)^F$ commute with the Hamiltonian, they preserve energy levels. So,

$$Q_i, (-1)^F : \mathcal{H}_{(m)} \to \mathcal{H}_{(m)}$$

Thus, each $\mathcal{H}_{(m)}$ is ${}_2$-graded

$$\mathcal{H}_{(m)} = \mathcal{H}^B_{(m)} \oplus \mathcal{H}^F_{(m)}$$

with $Q_i : \mathcal{H}^B_{(m)} \to \mathcal{H}^F_{(m)}$ and vice versa.

For nonzero energies, the number of bosonic and fermionic states in an energy level are equal. This is a particularly interesting feature of supersymmetric quantum mechanics.

Theorem 1.1. $\mathcal{H}^B_{(m)}$ is isomorphic to $\mathcal{H}^F_{(m)}$ for $m \neq 0$.

Proof. Let $|b\rangle \in \mathcal{H}^B_{(m)}$ be a bosonic state with nonzero energy E_m. Define the fermionic state $|f\rangle = \frac{1}{\sqrt{E_m}} Q_i |b\rangle$. Then,

$$Q_i|b\rangle = \sqrt{E_m}|f\rangle \text{ and } Q_i|f\rangle = 2\sqrt{E_m}|b\rangle$$

So, Q_i is invertible with inverse $Q_i^{-1} = \frac{1}{2E_m} Q_i$. This defines an isomorphism

$$\mathcal{H}^B_{(m)} \cong \mathcal{H}^F_{(m)}$$

□

We should note that $\mathcal{H}^B_{(0)}$ is not necessarily isomorphic to $\mathcal{H}^F_{(0)}$. So, even though nonzero energies yield isomorphic pairings between bosons and fermions, this is not necessarily the case in the zero energy case. Let's see why this is the case.

Proposition 1.3. Let $\alpha \in \mathcal{H}$ be arbitrary. Then, $H|\alpha\rangle = 0$ if and only if $Q_i|\alpha\rangle = 0$ for all $i = 1, ..., n$.

Proof. Suppose $Q_i|\alpha\rangle = 0$. Then, $H|\alpha\rangle = \frac{1}{2} Q_i^2 |\alpha\rangle = 0$.

Now, suppose $H|\alpha\rangle = 0$. Then,

$$0 = \langle\alpha|H|\alpha\rangle = \frac{1}{2}\langle\alpha|Q_i^2|\alpha\rangle = \frac{1}{2}(Q_i|\alpha\rangle)^\dagger(Q_i|\alpha\rangle) = \frac{1}{2}|Q_i|\alpha\rangle|^2$$

So, $Q_i|\alpha\rangle = 0$. □

Thus, Q_i is not generally invertible on $\mathcal{H}^B_{(0)}$ and $\mathcal{H}^F_{(0)}$, and $\mathcal{H}^B_{(0)}$ is not generally isomorphic to $\mathcal{H}^F_{(0)}$.

1.3 The Witten Index

The relative number of bosons to fermions at zero energy is a peculiar invariant. How? Well, suppose we promoted one boson in $\mathcal{H}^B_{(m)}$ to a higher energy state in $\mathcal{H}^B_{(M)}$. Theorem 1.1 requires that we promote a corresponding fermion to $\mathcal{H}^F_{(M)}$. So, the relative number of independent bosonic states and independent fermionic states in any energy subspace of $\mathcal{H}$ is invariant under such state-promotions. More precisely, in $\mathcal{H}_{(m)}$, this number is

$$n_m \equiv \dim(\mathcal{H}^B_{(m)}) - \dim(\mathcal{H}^F_{(m)})$$

In nonzero energy subspaces $\mathcal{H}_{(m\neq 0)}$, $n_m = 0$ by Theorem 1.1. However, n_0 could be nonzero in the zero energy subspace $\mathcal{H}_{(0)}$, but must remain invariant under state-promotions (deformations of the Hamiltonian). We call n_0 the **Witten index** of the supersymmetric quantum system.

The Witten index is closely related to the trace of the grading operator. Since the Hilbert space can be written as a decomposition of fermionic and bosonic subspaces we can represent the state in $\mathcal{H}$ as column vectors

$$\psi = \begin{pmatrix} \psi^B \\ \psi^F \end{pmatrix}$$

with $\psi^B \in \mathcal{H}^B$ and $\psi^F = \mathcal{H}^F$. In this basis, the grading operator $(-1)^F$ takes the form of a block matrix

$$(-1)^F = \begin{pmatrix} I^B & 0 \\ 0 & -I^F \end{pmatrix}$$

where I^B is the identity on $\mathcal{H}^B$ and I^F is the identity on $\mathcal{H}^F$.

Proposition 1.4. $n_0 = Tr(-1)^F$.

Proof. From the form of $(-1)^F$ above,

$$\begin{aligned} Tr(-1)^F &= \sum_m (\dim(\mathcal{H}^B_{(m)}) - \dim(\mathcal{H}^F_{(m)})) \\ &= \dim(\mathcal{H}^B_{(0)}) - \dim(\mathcal{H}^F_{(0)}) \\ &= n_0 \end{aligned}$$

where the second equality follows from Theorem 1.1. □

We can also demonstrate the invariance of the Witten index under state-promotions by showing that it is invariant under continuous deformations of the Hamiltonian.

Theorem 1.2. $n_0 = Tr((-1)^F e^{-\beta H})$ for all $\beta \in \mathbb{R}^+$ (β is sometimes called the *inverse temperature*).

Proof. Recall that $E_0 = 0$. So,

$$\begin{aligned} Tr((-1)^F e^{-\beta H}) &= \sum_m (\dim(\mathcal{H}^B_{(m)}) e^{-\beta E_m} - \dim(\mathcal{H}^F_{(m)}) e^{-\beta E_m}) \\ &= \dim(\mathcal{H}^B_{(0)}) e^{-\beta E_0} - \dim(\mathcal{H}^F_{(0)}) e^{-\beta E_0} \\ &= \dim(\mathcal{H}^B_{(0)}) - \dim(\mathcal{H}^F_{(0)}) \\ &= n_0 \end{aligned}$$

□

Remark 1.2. $Tr((-1)^F e^{-\beta H})$ is sometimes called the **supertrace** of $e^{-\beta H}$, denoted $Tr_s(e^{-\beta H})$.

2 Index Theorems and the Witten Index

Now, we will examine the relationship between the Witten index for an $\mathcal{N} = 1$ sussupersymmetric quantum system (the nonlinear sigma model, in particular) and the Atiyah–Singer index theorem. Reference Chapters 1.1 and 2.3 in [1] for the mathematical background required to parse the Atiyah–Singer index theorem.

2.1 Index Theorems

Theorem 2.1 (Atiyah–Singer index theorem)**.** Let E be a vector bundle over M, a compact oriented even-dimensional manifold and let S be a canonically graded Clifford bundle over M with associated Dirac operator $D : C^\infty(S_+ \otimes E) \to C^\infty(S_- \otimes E)$. Then,

$$\operatorname{index}(D) = \operatorname{ch}(E)\hat{\mathcal{A}}(M)[M],$$

where $\operatorname{ch}(E)$ is the Chern character of E and $\hat{\mathcal{A}}(M)$ is what is called the $\hat{\mathcal{A}}$-*genus* of M.

Proof. For brevity, we omit the proof of this theorem. However, there are multiple ways to prove it and multiple textbooks dedicated to doing so. For example, Roe's *Elliptic Operators, Topology, and Asymptotic Methods* [2] proves the Atiyah–Singer theorem using the heat equation and an asymptotic expansion of the heat kernel. Bleecker and Booß-Bavnek's *Index Theory with Applications to Mathematics and Physics* [3] proves it using tools from topological K-theory. □

Remark 2.1. Theorem 2.1 is a special case of the full Atiyah–Singer index theorem, which extends beyond Dirac operators to an index theorem for elliptic operators [4].

Example 2.1 (Index theorem for chiral Dirac operators)**.** If M is a spin manifold, then

$$\operatorname{index}(D) = \hat{\mathcal{A}}(M)$$

Example 2.2 (Index theorem for de Rham operators)**.** Let M be a smooth compact n-dimensional manifold and let $\Omega(M)$ be the space of smooth differential forms on M. So,

$$\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$$

where $\Omega^k(M)$ is the space of smooth k-forms on M. If $D = d + d^*$ is the de Rham operator on $\Omega(M)$, then the index of D is the Euler characteristic of M in the topological sense

$$\operatorname{index}(d + d^*) = \chi(M)$$

2.2 $\mathcal{N}=1$ Supersymmetric Nonlinear Sigma Model

Let M be a compact even-dimensional spin manifold with $\dim(M)=n$. Let

$$\mathcal{H}=C^\infty(M;\mathcal{S}_d)$$

be smooth sections of a spinor bundle over M with structure group $\mathrm{Spin}(d)$. Equip $\mathcal{H}$ with the L^2 inner product

$$\langle s,s'\rangle=\int_M\langle s(p),s'(p)\rangle\mathrm{vol}$$

where $\mathrm{vol}\in\Omega^n(M)$ is the volume form on M. Then, $\mathcal{H}=C^\infty(M;\mathcal{S}_d)$ is a Hilbert space.

Let ∇ be the corresponding spin connection and let γ^μ be the $2^{n/2}\times 2^{n/2}$ higher-dimensional gamma matrices. They satisfy the relation

$$\{\gamma^\mu,\gamma^\nu\}=2\delta^{\mu\nu}$$

Then, define the supercharge Q and Hamiltonian H as

$$Q=\frac{-i}{\sqrt{2}}\gamma^\mu\nabla_\mu=\frac{-i}{\sqrt{2}}\not{\nabla}:C^\infty(M;\mathcal{S}_d)\to C^\infty(M;\mathcal{S}_d)$$

$$H=\frac{Q^2}{2}=-\frac{1}{4}\not{\nabla}^2:C^\infty(M;\mathcal{S}_d)\to C^\infty(M;\mathcal{S}_d)$$

Finally, define the grading operator

$$(-1)^F=\gamma_\chi\equiv i^n\gamma^1...\gamma^{2n}:C^\infty(M;\mathcal{S}_d)\to C^\infty(M;\mathcal{S}_d)$$

We call γ_χ the **chirality operator**.

One can then check that

$$\mathcal{Q}\equiv(\mathcal{H},(-1)^F,H,Q)=\left(C^\infty(M;\mathcal{S}_d),\gamma_\chi,-\frac{1}{4}\not{\nabla}^2,-\frac{i}{\sqrt{2}}\not{\nabla}\right)$$

is an $\mathcal{N}=1$ supersymmetric quantum system, called the $\mathcal{N}=1$ **supersymmetric nonlinear sigma model**.

Remark 2.2. This supersymmetric quantum system may seem a little contrived. See section 5.1 of [5] for a better motivated particle physics-based approach to this system. In [5], the author builds $\mathcal{Q}$ from the $\mathcal{N}=1$ nonlinear sigma model Lagrangian.

Theorem 2.2. The Witten index of the $\mathcal{N}=1$ superymmetric quantum system $\mathcal{Q}$ defined above is equal to the $\hat{\mathcal{A}}$-genus $\hat{\mathcal{A}}(M)$.

Proof. M is an even-dimensional spin-manifold with associated spinor bundle $C^\infty(M; \mathcal{S}_d)$. The supercharge $Q = -\frac{i}{\sqrt{2}}\nabla\!\!\!/$ is a Dirac operator. So, by example 2.1,

$$\text{index}(Q) = \hat{\mathcal{A}}(M)$$

Additionally, by the McKean–Singer formula in example **??**,

$$\text{index}(Q) = Tr_s(e^{-\frac{t}{2}Q^2}) = Tr_s(e^{-\beta Q^2})$$

for $t \in \mathbb{R}^+$ and $\beta = \frac{t}{2}$. But, we know from theorem 1.2 and remark 1.2 that the Witten index n_0 satisfies

$$n_0 = Tr_s(e^{-\beta Q^2})$$

Thus,

$$\boxed{n_0 = \text{index}(Q) = \hat{\mathcal{A}}(M)}$$

□

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