Calculating Geometric Series Using Game Show Taxes

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Abstract

Canonical calculations of geometric series usually require clever algebraic manipulations of partial sums [1]. Such manipulations are not intuitive to everyone because not everyone knows algebra. However, everyone pays taxes. In 1789, in a letter to Jean-Baptiste Le Roy, Benjamin Franklin said, "in this world nothing can be said to be certain, except death and taxes". In this note, I use a game show tax model to calculate the geometric series $1 + 1/2 + 1/4 + 1/8 + \ldots$, without requiring the use of algebra. I also use this tax model to calculate more general geometric series.

1 The Tax Problem

Whenever I see people win prizes on game shows, like *Survivor* or *Let's Make a Deal*, I can't help but wonder about the tax burden they face on their winnings. If you win a sailboat on a game show, then you have to pay some percentage of its value in taxes. What if you don't have that much cash on hand? You might be forced to sell the boat at a price lower than its value just to pay the tax value of your prize. Sad.

Let us suppose that the game show's producers don't want the tax burden you incur to compel your loss of your prize. To solve your tax issue, they offer to award you with the cash value of taxes you have to pay. This is cash is also a part of your prize, and is taxable itself. How much cash should you be awarded to cover the tax on your total prize, in such a way that you don't have to sell the sailboat? There are two ways to answer this – let's examine both of them.

1.1 An Iterative Solution

Suppose the tax on every dollar you win is 50 cents. So, the tax rate is $\frac{1}{2}$. Suppose also that you win a sailboat worth 1 dollar.

After winning this prize, you are immediately overcome with grief because you need to pay a $\frac{1}{2}$ dollar tax on your sailboat. You approach the producers and ask them to give you $\frac{1}{2}$ a dollar. They give it to you. So, you now have a prize worth

$$1+\frac{1}{2}.$$

However, you realize that you have to pay a tax on the $\frac{1}{2}$ dollar cash you received from the producers. That tax amounts to $\frac{1}{4}$ of a dollar. So, you approach the producers and ask them for $\frac{1}{4}$ of a dollar. Your prize is now worth

$$1+\frac{1}{2}+\frac{1}{4}.$$

You once again realize you have to pay a tax of $\frac{1}{8}$ of a dollar on the $\frac{1}{4}$ dollar you received. This process repeats ad infinitum. Thus, the total prize is worth

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

This is complicated – you can't constantly go back to the producers to ask for more money. You have a life, after all. We need to devise a calculation by which we ask for all the cash at once.

1.2 A Direct Solution

You want to ask for some cash x from the producers to exactly cover the tax on your total prize worth $1+x$. The tax rate is $\frac{1}{2}$ — so you have to pay $\frac{1+x}{2}$ in taxes. If x must cover this exactly, we set

$$\frac{1+x}{2}=x.$$

Solving, we get $x=1$. So, after you ask for 1 dollar, your total prize is worth 2 dollars.

Remark 1.1. You could have just guessed that the producers should award you with 1 dollar in cash because then you would have to pay a tax of 1 dollar on your total prize worth 2 dollars. The algebra above was not incredibly necessary for this simple calculation, but it will illuminate a more general calculation in the subsequent section.

The 1/2 Geometric Series

Combining the total prize calculated in both solutions yields

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=2.$$

Succinctly,

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=2.$$

Remark 1.2. By Remark 1.1, we have calculated this geometric series without the need for algebra. I hope this makes it more universally intuitive than canonical calculations based on algebraic manipulations of partial sums [1].

2 General Tax Rates and General Geometric Series

A series $\sum_{n=0}^{\infty} r^n$ is a geometric series with rate $r \in [0, 1)$. We can use our game show tax example to calculate the value of this geometric series.

Instead of a tax rate of $\frac{1}{2}$, suppose we tax contestants at a rate $r \in [0, 1)$. Then, the iterative method for calculating the total prize necessary to avoid paying tax on the sailboat yields a total prize worth

$$1 + r + r^2 + r^3 + \ldots$$

The direct method yields

$$r(1 + x) = x \implies x = \frac{r}{1 - r},$$

suggesting a total prize worth

$$1 + \frac{r}{1 - r} = \frac{1}{1 - r}.$$

A General Geometric Series

Combining the total prize calculated using both methods yields the desired calculation. For $0 \le r < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

Remark. This result holds for a larger range of r — namely $-1 < r < 1$.

References

[1] Wikipedia. *1/2 + 1/4 + 1/8 + 1/16 + ...* URL: https://en.wikipedia.org/wiki/1/2_%2B_1/4_%2B_1/8_%2B_1/16_%2B_%E2%8B%AF#Proof.