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Bridging Partitions Between Statistical Mechanics, Quantum Mechanics, and Black Holes

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Arjun Srinivasan Kudinoor

Abstract: Partition functions are ubiquitous objects of study in statistical physics, quantum mechanics, and quantum field theory. The appearance and use of partition functions in each of these sectors of physics motivates the thesis of this article: to translate between descriptions of partition functions in each of these sectors. To do so, I first introduce a path integral formulation of quantum mechanics. I then use this to show that the partition function of a canonical ensemble of microstates in thermal equilibrium at inverse temperature β is equivalent to the partition function of an ensemble of quantum mechanical fields that are periodic in time, with period $-i\beta\hbar$. Finally, I describe how this equivalence can be used to calculate the Hawking temperature of Schwarzschild black holes.

Contents

Contents

1 From Statistical Mechanics to Quantum Mechanics

Suppose we have a canonical ensemble of microstates in thermal equilibrium with a heat bath at temperature $k_B T = 1/\beta$. The probability that this system will be in a state with energy E_i is given by the Boltzmann weight

$$p(E_i) = \frac{e^{-\beta E_i}}{Z}, \tag{1.1}$$

where $\beta = 1/(k_B T)$ is the temperature of the system (with dimensions of the inverse energy) and Z is the partition function

$$Z = \sum_i p(E_i) = \sum_i e^{-\beta E_i}. \tag{1.2}$$

Here, i indexes the microstates of the system. Suppose further that each energy E_i of the system is an eigenvalues of the Hamilton H, with some corresponding set of eigenvectors. Label each (orthonormal) eigenvector of H by $|n\rangle$. Then, the partition function becomes

$$Z = \text{Tr}\left(e^{-\beta H}\right) = \sum_n \langle n|e^{-\beta H}|n\rangle. \tag{1.3}$$

Suppose also that we have a continuous set of position eigenstates $|q\rangle$. Then, we know that $\int dq |q\rangle\langle q| = \mathbb{1}$. Inserting this into Eq.1.3, we get

$$Z = \int dq \sum_n \langle n|q\rangle\langle q|e^{-\beta H}|n\rangle \tag{1.4}$$

$$= \int dq \sum_n \langle q|e^{-\beta H}|n\rangle\langle n|q\rangle \tag{1.5}$$

$$= \int dq \langle q|e^{-\beta H}\left(\sum_n |n\rangle\langle n|\right)|q\rangle \tag{1.6}$$

$$= \int dq \langle q|e^{-\beta H}\mathbb{1}|q\rangle \tag{1.7}$$

$$= \int dq \langle q|e^{-\beta H}|q\rangle \tag{1.8}$$

We could have deduced that $Z = \int dq\langle q|e^{-\beta H}|q\rangle$ by noting that the partition function is simply the trace of the operator $e^{-\beta H}$, so it does not matter if we take this trace with respect to the energy basis $\{|n\rangle\}$ or the position basis $\{|q\rangle\}$.

With some foresight, let us identify β with a variable t, defined by $\beta = it/\hbar$. Then, if we identify t as a time-coordinate, then the operator $e^{-\beta H} = e^{-itH/\hbar}$ is the time-evolution operator from quantum mechanics! We can then write the partition function as

$$Z = \int dq \langle q|e^{-itH/\hbar}|q\rangle \tag{1.9}$$

This expression looks like an integral over all paths q such that $|q\rangle$ is an eigenvector of H, i.e. $H|q\rangle = \lambda_q|q\rangle$. In the next section, we use the language of propagators and Feynman path integrals to make the path integral description of the partition function more explicit.

2 The Path Integral

The time evolution of a quantum state $|\psi(t)\rangle$ from a time t' to another time t is given by $|\psi(t')\rangle = e^{-i(t-t')H/\hbar}|\psi(t')\rangle$. In the position representation,

$$\langle q|\psi(t)\rangle = \langle q|e^{-i(t-t')H/\hbar}|\psi(t')\rangle = \int dq' \langle q|e^{-i(t-t')H/\hbar}|q'\rangle\langle q'|\psi(t')\rangle, \tag{2.1}$$

where in the last equality we simply inserted the identity operator $\int dq'|q\rangle\langle q'|$. We can rewrite this as $\langle q|\psi(t)\rangle = \int dq' K(q,t;q',t')\langle q'|\psi(t')\rangle$, where we have defined the "propagator"

$$K(q,t;q',t') \equiv \langle q|e^{-i(t-t')H/\hbar}|q'\rangle. \tag{2.2}$$

The propagator $K(q,t;q',t')$ is interpreted as a transition amplitude, which when squared yields the probability of the system to transition from state $|q'\rangle$ at time t' to $|q\rangle$ at time t. To see this, define $|q,t\rangle \equiv e^{iHt/\hbar}|q\rangle$. Then,

$$\begin{aligned} K(q,t;q',t') &= \langle q|e^{-i(t-t')H/\hbar}|q'\rangle \\ &= \langle q|e^{-itH/\hbar}e^{i'tH/\hbar}|q'\rangle \\ &= \left(e^{itH/\hbar}|q\rangle\right)^\dagger \left(e^{i'tH/\hbar}|q'\rangle\right) \\ &= (|q,t\rangle)^\dagger \left(|q',t'\rangle\right) \\ &= \langle q,t|q',t'\rangle. \end{aligned}$$

Squaring the final expression above yields the probability of transition from $|q',t'\rangle$ to $|q,t\rangle$, as desired. Now, let's evaluate the propagator. First, we partition the time interval $[t',t]$

into N subintervals of each of size $\Delta t = (t - t')/N$. Let $\{t_k\}_{k=0}^{k=N}$ define the partition, where $t_0 = t'$, $t_N = t$, and $t_k = t' + k\Delta t$. Then,

$$K(q, t; q', t') = \langle q|e^{-i(t-t')H/\hbar}|q'\rangle = \langle q|e^{-iN\Delta tH/\hbar}|q'\rangle = \langle q| \left(\prod_{k=0}^{N} e^{-i\Delta tH/\hbar}\right)|q'\rangle \tag{2.3}$$

Trading concision for understanding, we expand

$$\begin{aligned}K(q, t; q', t') &= \langle q|e^{-i\Delta tH/\hbar} \cdots e^{-i\Delta tH/\hbar}|q'\rangle \\ &= \int dq_1 dq_2 \cdots dq_{N-1} \langle q|e^{-i\Delta tH/\hbar}|q_{N-1}\rangle\langle q_{N-1}|e^{-i\Delta tH/\hbar}|q_{N-2}\rangle \cdots \langle q_1|e^{-i\Delta tH/\hbar}|q'\rangle \\ &= \int \left(\prod_{k=1}^{N-1} dq_k\right) \langle q, t|q_{N-1}, t_{N-1}\rangle\langle q_{N-1}, t_{N-1}|q_{N-2}, t_{N-2}\rangle \cdots \langle q_1, t_1|q', t'\rangle\end{aligned}$$

In the second line of the above set of equations, we inserted N copies of the identity operator $\mathbb{1} = \int dq_k|q_k\rangle\langle q_k|$ between each of the time-evolution operators. In the third line, we identified each of the factors in the integrand as transition amplitudes for the system to transition from state $|q_k\rangle$ at time t_k to state $|q_{k+1}\rangle$ at time t_{k+1}. How do we interpret this? At each time step t_k, we allow for the system to be in any state $|q_k\rangle$. So, we integrate over all possible values q_k that the system could be in at that time step. We repeat this for each time step and get an integral over all possible paths that the system can take in this discretely partitioned time interval $[t', t]$.

Now, let us calculate the transition amplitude $\langle q_k, t_k|q_{k-1}, t_{k-1}\rangle$ for a single subinterval of time. To make our lives easier, we will enforce that the number of partitions N is asymptotically large, i.e. our discretization of the time interval $[t', t]$ should be infinitesimal. As $N \to \infty$, $\Delta t \to 0$. So, we can Taylor expand the time-evolution operator $e^{-i\Delta tH\hbar}$ around $\Delta t = 0$. Then,

$$\begin{aligned}\langle q_k, t_k|q_{k-1}, t_{k-1}\rangle &= \langle q_k|e^{-i\Delta tH/\hbar}|q_{k-1}\rangle \\ &= \langle q_k|\left(1 - \frac{i}{\hbar}\Delta tH + \mathcal{O}(\Delta t^2)\right)|q_{k-1}\rangle \\ &= \langle q_k|q_{k-1}\rangle - \frac{i}{\hbar}\Delta t\langle q_k|H|q_{k-1}\rangle + \mathcal{O}(\Delta t^2) \\ &= \delta(q_k - q_{k-1}) - \frac{i}{\hbar}\Delta t\langle q_k|H|q_{k-1}\rangle + \mathcal{O}(\Delta t^2),\end{aligned}$$

where in the last line we used the fact that the states $\{|q_k\rangle\}$ are orthonormal. To evaluate $\langle q_k|H|q_{k-1}\rangle$, we recall that at each time step t_k, the Hamiltonian $H = p^2/(2m) + V(q)$ has momentum eigenstates $\{|p_k\rangle\}$ such that $p|p_k\rangle = p_k|p_k\rangle$. So,

$$H|p_k\rangle = \left(\frac{p^2}{2m} + V(q)\right)|p_k\rangle = \left(\frac{p_k^2}{2m} + V(q)\right)|p_k\rangle. \tag{2.4}$$

Additionally, recall that

$$\langle p|q\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipq/\hbar}. \tag{2.5}$$

We insert the identity $\mathbb{1} = \int dp_k |p_k\rangle\langle p_k|$ into $\langle q_k|H|q_{k-1}\rangle$ to get

$$\begin{aligned}
\langle q_k|H|q_{k-1}\rangle &= \int dp_k \langle q_k|H|p_k\rangle\langle p_k|q_{k-1}\rangle \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dp_k e^{-ip_k q_{k-1}/\hbar} \langle q_k|H|p_k\rangle\langle p_k| \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dp_k e^{-ip_k q_{k-1}/\hbar} \langle q_k| \left(\frac{p^2}{2m} + V(q)\right) |p_k\rangle \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dp_k e^{-ip_k q_{k-1}/\hbar} \langle q_k|p_k\rangle \left(\frac{p_k^2}{2m} + V(q_k)\right) \\
&= \frac{1}{2\pi\hbar} \int dp_k e^{ip_k(q_k - q_{k-1})/\hbar} \left(\frac{p_k^2}{2m} + V(q_k)\right)
\end{aligned}$$

Substituting this back into the expression for $\langle q_k, t_k|q_{k-1}, t_{k-1}\rangle$, we see that

$$\begin{aligned}
\langle q_k, t_k|q_{k-1}, t_{k-1}\rangle &= \frac{1}{2\pi\hbar} \int dp_k e^{ip_k(q_k - q_{k-1})/\hbar} \left(1 - \frac{i}{\hbar}\Delta t \left(\frac{p_k^2}{2m} + V(q_k)\right) + \mathcal{O}(\Delta t^2)\right) \\
&= \frac{1}{2\pi\hbar} \int dp_k e^{ip_k(q_k - q_{k-1})/\hbar} e^{-i\Delta t/\hbar\left(\frac{p_k^2}{2m} + V(q_k)\right)} \\
&= \frac{1}{2\pi\hbar} \int dp_k e^{i\Delta t/\hbar\left(p_k \frac{(q_k - q_{k-1})}{\Delta t} - \frac{p_k^2}{2m} - V(q_k)\right)}
\end{aligned}$$

We now recall that we took $N \to \infty$, and we also assume that q changes continuously. Then, $(q_k - q_{k-1})/\Delta t \to \dot{q_k}$. Thus,

$$\langle q_k, t_k|q_{k-1}, t_{k-1}\rangle = \frac{1}{2\pi\hbar} \int dp_k e^{i\Delta t/\hbar\left(p_k \dot{q_k} - \frac{p_k^2}{2m} - V(q_k)\right)} \tag{2.6}$$

We evaluate the integral over the terms containing p_k first.

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp_k e^{i\Delta t/\hbar\left(p_k \dot{q_k} - \frac{p_k^2}{2m}\right)} = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} e^{\frac{i}{\hbar}\Delta t \frac{m\dot{q_k}^2}{2}}. \tag{2.7}$$

Thus,

$$\langle q_k, t_k|q_{k-1}, t_{k-1}\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} e^{\frac{i}{\hbar}\Delta t\left(\frac{m\dot{q_k}^2}{2} - V(q_k)\right)} = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} e^{\frac{i}{\hbar}\Delta t L(q_k, \dot{q_k})}, \tag{2.8}$$

where $L(q, \dot{q})$ is the Lagrangian of our system, given by $L = p\dot{q} - H$.

We now substitute this into our expression for the propagator to see

$$K(q, t; q', t') = \int \left(\prod_{k=1}^{N-1} dq_k\right) \langle q, t|q_{N-1}, t_{N-1}\rangle\langle q_{N-1}, t_{N-1}|q_{N-2}, t_{N-2}\rangle \cdots \langle q_1, t_1|q', t'\rangle$$

$$= \int \left(\prod_{k=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} dq_k \right) e^{\frac{i}{\hbar}\Delta t L(q,\dot{q})} e^{\frac{i}{\hbar}\Delta t L(q_{N-1},\dot{q}_{N-1})} \dots e^{\frac{i}{\hbar}\Delta t L(q_1,\dot{q}_1)}$$

$$= \int \left(\prod_{k=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} dq_k \right) e^{\frac{i}{\hbar}\sum_{k=1}^{N} \Delta t L(q_k,\dot{q_k})}$$

$$= \int \left(\prod_{k=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} dq_k \right) e^{\frac{i}{\hbar}\int dt L},$$

where in the last line we took $\Delta \to 0$ and identified the sum with an integral over t. Define the "path integral measure"

$$\int Dq(t) \equiv \lim_{N\to\infty} \int \left(\prod_{k=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} dq_k \right) dq' \tag{2.9}$$

Thus, the propagator can be written as an integral over all *field configurations* $q(t)$ that our system can assume.

$$\boxed{\int dq' K(q,t;q',t') = \int Dq(t) e^{\frac{i}{\hbar}\int dt L}} \tag{2.10}$$

3 From Quantum Mechanics to Statistical Mechanics

Finally, let us connect this path integral formulation to our expression of the partition function from statistical mechanics. Suppose that we require that all paths $q(t)$ of our system be periodic in t. In particular, for $\mathcal{T} \equiv -i\beta\hbar$, we require that $q(\mathcal{T}) = q(0)$. Here, $\beta = 1/(k_B T)$ is the inverse temperature of the system. Then,

$$\begin{aligned} Z &= \int dq' \langle q'| e^{-\beta H} |q'\rangle \\ &= \int dq' \langle q'| e^{-i\mathcal{T}H/\hbar} |q'\rangle \\ &= \int dq' K(q',\mathcal{T};q'0) \\ &= \int_{q(0)=q(\mathcal{T})} Dq(t) e^{\frac{i}{\hbar}\int dt L}. \end{aligned}$$

Therefore,

$$\boxed{Z = \int_{q(0)=q(-i\beta\hbar)} Dq(t) e^{\frac{i}{\hbar}\int dt L}.} \tag{3.1}$$

The fact that a statistical ensemble in thermal equilibrium corresponds to a theory of quantum mechanical fields $q(t)$ that are periodic with period $-i\beta\hbar$ has some really cool consequences. We explore one such consequence below – deriving the Hawking temperature of a black hole!

4 Black Hole Temperature From Quantum Statistical Mechanics

Suppose we have a quantum matter system in thermal equilibrium in Minkowski spacetime $\mathbb{R}^{1,3}$.

$$ds^2 = -dt^2 + d\vec{x}^2. \tag{4.1}$$

We can write the partition function of this system as

$$Z = \int_{q(0)=q(-i\beta\hbar)} Dq(t)e^{iS/\hbar}, \tag{4.2}$$

where we have identified $S = \int dtL$ as the action functional of our system. To remove some annoying factors of i in our expression of the action, let us move to a coordinate system with imaginary time τ, i.e. $t \to -i\tau$. Then, the action transforms as $iS \to -S_E$, where S_E is the "Euclidean action". So, our partition function becomes

$$\int_{q(0)=q(\beta\hbar)} Dq(\tau)e^{-S_E[q(\tau)]}, \tag{4.3}$$

where $S_E[q]$ and q are defined on Euclidean space $S^1 \times \mathbb{R}^3$

$$ds_E^2 = d\tau^2 + d\vec{x}^2, \quad \tau \sim \tau + \beta\hbar. \tag{4.4}$$

In the second expression above, the periodicity of τ is denoted $\tau \sim \tau + \beta\hbar$.

We note that this metric describes a *regular* Euclidean manifold and β is a free parameter that we can vary. Therefore, our quantum system can freely assume any temperature in this flat spacetime. However, let us now consider a region of spacetime that is very much not flat, like near a Schwarzschild black hole.

The metric of spacetime in the presence of a Schwarzschild black hole of mass M is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2, \tag{4.5}$$

where $f(r) = 1 - \frac{r_s}{r}$, where $r_s = 2G_N M$ is the Schwarzschild radius of the black hole. Note that we are assuming $c = 1$. Moving to imaginary time $t \to -i\tau$, the Euclidean Schwarzschild metric is

$$ds_E^2 = f(r)d\tau^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2. \tag{4.6}$$

Let's try to understand the regularity of this Euclidean manifold by looking at the metric close to the event horizon, i.e. close to $r = r_s$. Close to the horizon, the proper distance ρ for $r \gtrsim r_s$ is given by

$$d\rho = \frac{dr}{\sqrt{f}} = \frac{dr}{\sqrt{f(r_s) + (r - r_s)f'(r_s) + \mathcal{O}((r - r_s)^2)}} = \frac{dr}{\sqrt{(r - r_s)f'(r_s) + \mathcal{O}((r - r_s)^2)}}, \tag{4.7}$$

where we Taylor expanded $f(r)$ around r_s and noted that $f(r_s) = 0$. Solving this for ρ,

$$\rho(r) = \frac{2}{\sqrt{f'(r_s)}}\sqrt{r - r_s} + \dots . \tag{4.8}$$

To leading order in the expansion of $(r - r_s)$, we can write the Euclidean Schwarzschild metric near the event horizon $r \gtrsim r_s$ as

$$ds_E^2 = \kappa^2 \rho^2 d\tau^2 + d\rho^2 + r_s^2 d\Omega_2^2 = \rho^2 d\theta^2 + d\rho^2 + r_s^2 d\Omega_2^2, \tag{4.9}$$

where $\theta \equiv \kappa\tau$ and $\kappa = \frac{1}{4G_N M}$ is the surface gravity of the black hole. The global structure of the (ρ, θ) space depends on the periodicity of θ. If θ has periodicity 2π, then the (ρ, θ) space is $\mathbb{R}^2$. Otherwise, this space has a concical singularity at the event horizon, i.e. at $\rho = 0$. Thus, regularity of this spacetime manifold requires that

$$\theta \sim \theta + 2\pi \implies \tau \sim \tau + \frac{2\pi}{\kappa}. \tag{4.10}$$

Now, suppose we have a quantum matter system in thermal equilibrium in this black hole geometry. Then, Since the quantum fields $q(\tau)$ that describe this system must be periodic in τ, we have an additional constraint on the periodicity of τ, namely $\tau \sim \tau + \beta\hbar$. So,

$$\beta\hbar = \frac{2\pi}{\kappa} \implies T = \frac{\hbar\kappa}{2\pi k_B} = \frac{\hbar}{8\pi G_N M k_B}. \tag{4.11}$$

Restoring factors of c, we find that

$$\boxed{T_H = \frac{\hbar c^3}{8\pi G_N M k_B}}. \tag{4.12}$$

The subscript H in T_H is to label T_H as the "Hawking temperature" of the black hole.

Recall that the time-coordinate t is the proper time of an observer asymptotically far away from the black hole at $r = \infty$. To such an observer, a quantum matter system in the black hole geometry can be in thermal equilibrium only at a single temperature, namely T_H. This must also be the temperature of the black hole, since otherwise the total system would not be in thermal equilibrium.

To learn more, about calculating black hole temperatures using regularization techniques for Euclidean manifolds, check out Hong Liu's lectures on String Theory and Holographic Duality at Ref.[1].

References

[1] H. Liu, *String theory and holographic duality*, 2014. Lecture 3, Available at https://ocw.mit.edu/courses/8-821-string-theory-and-holographic-duality-fall-2014/pages/lecture-notes/.