1 Black-Scholes option pricing formula

As we saw previously in lecture, the option price, C_0, of certain kinds of derivatives of stock (such as a European call option), with expiration date $t = T$, when using the binomial lattice model (BLM), turns out to be a discounted expected value of payoff at time $t = T$, C_T, under the so-called "risk neutral" probability $p*$;

$$C_0 = (1+r)^{-T} E^*(C_T),$$

where $r > 0$ is the interest rate, and E^* denotes expected value when the "up" probability for the stock is

$$p = p^* = \frac{1+r-d}{u-d},$$

where (non-arbitrage assumption)

$$0 < d < 1+r < u,$$

ensures that indeed $p^* \in (0,1)$.

This elegant result involves the fact that p^* is the unique value for $p \in (0,1)$ making the discounted stock price process $\{(1+r)^{-n}S_n : n \geq 0\}$ form a martingale, implying that on average, under p^*, the stock yields the same rate of return as the risk-free asset at rate r: $E(S_n) = S_0(1+r)^n$.

Our proof of this option pricing formula was based on a replication method: Find an alternative investment (portfolio) (α, β) of stock and risk-free asset, such that it yields (e.g., replicates) the same payoff as the option. Since they yield the same payoff, they must cost the same, and since the cost of (α, β) is simply $\alpha S_0 + \beta$, we are done; $C_0 = \alpha S_0 + \beta$. It then turned out that this cost could be re-written as $(1+r)^{-n}E^*(C_n)$.

This motivates how to price such derivatives when the stock pricing follows a geometric BM $S(t) = S_0 e^{\sigma B(t)+\mu t}$ for which the BLM was an approximation. Choosing the drift term μ of the underlying BM to be $\mu^* = r - \sigma^2/2$, results in $e^{-rt}S(t)$ forming a MG; $E^*(S(t)) = e^{rt}S_0$; on average, under μ^*, the stock yields the same rate of return as the risk-free asset at rate r. Following this logic, we obtain, for (say) a European call option (strike price K, expiration date T)

$$C_0 = e^{-rt}E^*(S(T) - K)^+,$$

where E^* denotes expectation when $\mu = \mu^*$.

Our purpose in these notes is to show that this is correct, but more generally to introduce the reader to the general theory of option pricing; the classic work attributed to Fisher Black, Myron S. Scholes and Robert C. Merton[1] in the 1970's and for which Scholes and Merton earned the Nobel Prize in Economics in 1997;
http://www.nobel.se/economics/laureates/1997/. We will do so by using SDE's and once again a replication method: we will construct a portfolio that changes over time (due to following an investment strategy that allows us to re-balance our investment continuously during $(0, T]$) in such a way that it replicates the option payoff.

[1]Robert C. Merton was an undergraduate student in Engineering at Columbia University, earning a BS in the 1960's in what was then the division of Applied Math and Operations Research.

1.1 The model for a European call option and beyond

We have two continuous time processes.

1. The stock price per share: $S(t) = S_0 e^{\sigma B(t)+\mu t}$; the SDE for this geometric BM is $dS(t) = \overline{\mu} S(t)dt + \sigma S(t)dB(t)$, where $\overline{\mu} = \mu + \sigma^2/2$.

2. The risk-free asset (bond (say)) per share: $b(t) = e^{rt}$, a deterministic function of t where $r > 0$ is the interest rate; the differential equation for this asset is $db(t) = rb(t)$. (We can view this asset as costing \$1 per share (always) and $b(t) = e^{rt}$ is the number of shares we get at time t if we bought one share at time 0.)

The payoff at expiration date $t = T$ of the option is a function of the stock price $S(T)$ at that time; $(S(T) - K)^+$, where $K \geq 0$ is the strike price.

We wish to construct a portfolio $(\alpha(t), \beta(t))$, denoting the number of shares of stock and risk-free asset we have at any time $t \in [0, T]$, the value of which matches the option payoff. At any given time $t \in [0, T]$ the value of our portfolio is $V(t) = \alpha(t)S(t) + \beta(t)b(t)$. Our objective is to buy some initial shares at time $t = 0$, (α_0, β_0), and then continuously readjust our portfolio in such a way that at time T we have $V(T) = (S(T) - K)^+$; for then we can conclude that the cost of the option is given by $C_0 = V(0) = \alpha_0 S_0 + \beta_0$.

An important observation: Let $C(t, S(t))$ denote the (apriori unknown) price of the option at time $t \in [0, T]$: Imagine that at time t you wish to sell the option that you bought at time 0; this price is what we denote by $C(t, S(t))$. We know that $C(T, S(T)) = (S(T) - K)^+$, and we want to find the value of $C_0 = C(0, S(0))$. But clearly, $C(t, S(t))$ is the same as the price of a European call option with the same strike price but expiration date $T - t$ and initial price $S(t)$. We see that it should hold that $V(t) = C(t, S(t))$ for all t not just for $t = T$. Thus we are looking for a deterministic *function* $C(t, x)$, denoting the cost of the option when the expiration date is $T - t$ and the current price is $x \geq 0$, $t \in [0, T]$. This implies that the stochastic processes, $\alpha(t)$ and $\beta(t)$ also can be expressed by deterministic functions, $f_\alpha(t, x)$ and $f_\beta(t, x)$, since at any time t their values only will depend on the values $T - t$ and $S(t)$; $\alpha(t) = f_\alpha(t, S(t))$ and $\beta(t) = f_\beta(t, S(t))$.

1.2 Self-financing

If we are to truly replicate our option investment via initially spending $\alpha_0 + \beta_0$ which is to be identical to the cost of the option, then we can not allow any cash flow (in or out) during $(0, T)$ for our portfolio readjustments, meaning, for example, that we can not buy new shares of assets with an outside source of cash along the way: any change in $V(t)$ must equal the profit or loss due to changes in the price of the two assets themselves. For example, if at time t we have $\alpha(t)S(t) = \$500$, then we can exchange $\alpha(t)/2$ shares of stock for \$250 worth of risk-free asset (so $V(t)$ does not change), but we can't just sell the $\alpha(t)/2$ shares and remove this money from the portfolio.

Formally, we call this a *self-financing* strategy, and it can be mathematically enforced by assuming that

$$dV(t) = \alpha(t)dS(t) + \beta(t)db(t), \text{ self-financing condition.} \tag{1}$$

Note that using Ito's formula on the Ito process V would yield several additional terms for $dV(t)$; the self-financing condition demands that they total 0 thus restricting the kind of process that V can be. In integral form, this condition is given by

$$V(t) = \alpha_0 S_0 + \beta_0 + \int_0^t \alpha(s)dS(s) + \int_0^t \beta(s)db(s).$$

1.3 Black-Scholes PDE

In the following, we generalize to allow for general derivative of the stock, not just European call options. Thus we let $f(t,x)$ denote the price of such a derivative at time t if $S(t) = x$. f plays the general role of our $C(t,x)$ above. Initial conditions would have to be specified for each specific derivative. For example, $f(t,0) = 0,\ t \in [0,T]$ and $f(T,x) = (x-K)^+,\ x \geq 0$ are the initial conditions for the European call option.

Theorem 1.1 (Black-Scholes Partial Differential Equation (PDE)) *Let $f(t,x)$ denote the price at time t of a derivative of stock (such as a European call option) when $S(t) = x$. Then f must satisfy the partial differential equation:*

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} rx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 x^2 = rf.$$

Before we prove this, note that the equation, hence the solution, does not depend on the drift μ of the underlying BM. Also note that if, for example, we chose the stock itself as our derivative so that $f(t,x) = x$, then indeed it satisfies the PDE. We could also choose the risk-free asset itself, $f(t,x) = e^{rt}$, and it too satisfies the PDE. But more importantly, it tells us that the price of our European call option (or any other derivative) MUST too satisfy the PDE, so together with the initial conditions $C(t,0) = 0,\ t \in [0,T]$ and $C(T,x) = (x-K)^+,\ x \geq 0$, we now have a way of solving for the prices.

Proof :[Black-Scholes PDE]
Using our differentials $dS(t)$ and $db(t)$ together with the self-financing condition yields

$$\begin{aligned} dV(t) &= \alpha(t)[\overline{\mu}S(t)dt + \sigma S(t)dB(t)] + \beta(t)rb(t)dt & (2) \\ &= (\alpha(t)\overline{\mu}S(t) + \beta(t)rb(t))dt + \alpha(t)\sigma S(t)dB(t). & (3) \end{aligned}$$

We are looking for a function $f(t,x)$ and portfolio factors $\alpha(t)$ and $\beta(t)$ such that for $V(t) = \alpha(t)S(t) + \beta(t)b(t)$, we also have $V(t) = f(t,S(t)),\ t \in [0,T]$

Using Ito's formula (Theorem 1.2 in Lecture Notes 7) on $V(t) = f(t,S(t))$ where $(dS(t))^2 = \sigma^2 S^2(t)dt$ denotes the quadratic variation term for geometric BM ($S(t)$ is an Ito process with $H(t) = \sigma S(t)$ and $K(t) = \overline{\mu}S(t)$):

$$\begin{aligned} df(t,S(t)) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dS(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dS(t))^2 & (4) \\ df(t,S(t)) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dS(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 S^2(t)dt & (5) \\ &= \Big[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 S^2(t) + \frac{\partial f}{\partial x}\overline{\mu}S(t)\Big]dt + \frac{\partial f}{\partial x}\sigma S(t)dB(t). & (6) \end{aligned}$$

Equating the $dB(t)$ coefficients from (3) and (6) yields

$$\alpha(t) = \frac{\partial f}{\partial x}(t, S(t)).$$

Similarly equating the two dt coefficients yields

$$\beta(t) = \frac{1}{rb(t)}\Big(\frac{\partial f}{\partial t}(t, S(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S(t))\sigma^2 S^2(t)\Big).$$

Finally, plugging in these values for $\alpha(t)$ and $\beta(t)$ while equating $\alpha(t)S(t)+\beta(t)b(t) = f(t, S(t))$ then replacing $S(t)$ with x yields the Black-Scholes PDE as was to be shown. ■

1.4 Solution to Black-Scholes PDE for the European call option

Except for some special cases, there is no analytical solution to the Black-Scholes PDE, but the European call option is such a special case and it is known as the famous

Theorem 1.2 (Black-Scholes option pricing formula) *The solution to the Black-Scholes PDE for the European call option with strike price K and expiration date T is given by $f(t, S) = C(t, S),\ t \in [0, T],\ S \geq 0$ where*

$$C(t, S) = S\Theta(c_1(t)) - Ke^{-r(T-t)}\Theta(c_2(t)),$$

where $\Theta(z) = P(Z \leq z)$ is the cdf for the unit normal $N(0, 1)$, and the constants $c_1(t)$ and $c_2(t)$ are given by

$$c_1(t) = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$c_2(t) = \frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} = c_1(t) - \sigma\sqrt{T - t}.$$

Note how time shows up as *remaining time* $T - t$, as makes sense. Also note that when $t = 0$ we get the initial price C_0 that we originally asked for :

The price of a European call option with strike price K and expiration date T is given by

$$C_0 = C_0(S_0, T, K, \sigma, r) = S_0\Theta(c_1) - Ke^{-rT}\Theta(c_2),$$

where

$$c_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$c_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

This price can be re-expressed as the discounted expected payoff of the option under the risk-neutral probability for the geometric BM: change μ to $\mu^* = r - \sigma^2/2$, and let $X^*(t) = \sigma B(t) + \mu^* t$ and $S^*(t) = S_0 e^{X^*(t)}$; $\{e^{-rt}S^*(t) : t \geq 0\}$ is then a Martingale and

$$C_0 = e^{-rT}E(S^*(T) - K)^+.$$

To prove this, one merely needs to compute the integral

$$e^{-rt}E(S^*(T)-K)^+ = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(S_0e^{\sigma\sqrt{T}x+\mu^*T}-K)^+e^{-\frac{x^2}{2}}dx,$$

and see that indeed it yields C_0 above. Without going thru the detailed calculation showing this, let us study the second piece in the price, $Ke^{-rT}\Theta(c_2)$. Note that

$$\begin{aligned}
\Theta(c_2) &= P(Z \le c_2)\\
&= P(\sigma\sqrt{T}Z-(r-\sigma^2/2)T \le \ln(S_0/K))\\
&= P(\sigma\sqrt{T}Z+(r-\sigma^2/2)T \ge \ln(K/S_0))\\
&= P(e^{\sigma\sqrt{T}Z+(r-\sigma^2/2)T} \ge K/S_0)\\
&= P(S_0e^{\sigma\sqrt{T}Z+(r-\sigma^2/2)T} \ge K)\\
&= P(S^*(T) \ge K)\\
&= P\Big((S^*(T)-K)^+ > 0\Big).
\end{aligned}$$

In other words, $\Theta(c_2)$ is the probability (under the risk-neutral probability), that the option will be excercised. Thus $Ke^{-rT}\Theta(c_2)$ can be realized as the price of K shares of a digital option that pays \$1 if $S(T) > K$ and 0 otherwise: The payoff (at time T) for one share is $I\{S(T) > K\}$ and its discounted expected value under the risk-neutral probability is $e^{-rT}\Theta(c_2)$; that is its price.