

Production

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GR6211 - Microeconomic Analysis 1

- Up until now we have dealt exclusively with one type of economic agent: the consumer
 - Defined by a set of preferences
- We are now going to deal (very quickly) with the other basic type of economic agent: the firm
- As much as possible I am going to try to convince you that you already know how to deal with the problem of a firm
 - Familiar problems with a new name

- 1 The Production Vector
- 2 Cost Minimization and the Cost Function
- 3 Profit Maximization

- The reason we really need firms for economic analysis is that so far we are missing something important
 - Where does stuff come from for consumers to consume?
- We could assume that it is just lying around
 - As in an endowment economy
- But this misses out the fact that there are lots of economic actors that produce the stuff that consumers buy
 - i.e. firms
- Moreover, these firms don't just sell stuff that they were endowed with
- They **convert** inputs into outputs
 - e.g. a baker converts labor and flour into bread
 - a university converts students and professors into knowledge (?)

- So, while in principle a firm could be characterized by lots of things
 - Its organization
 - Its motivation
 - Who owns it
- We will define it by its ability to transform things from one type to another
 - i.e. its technology.

- Let's imagine we are in an L commodity world
- A production vector is just an L length vector which describes the net output of each good
 - e.g.

$$\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

- Would mean **using** 3 units of good 1 to produce 2 units of good 2 and 1 unit of good 3

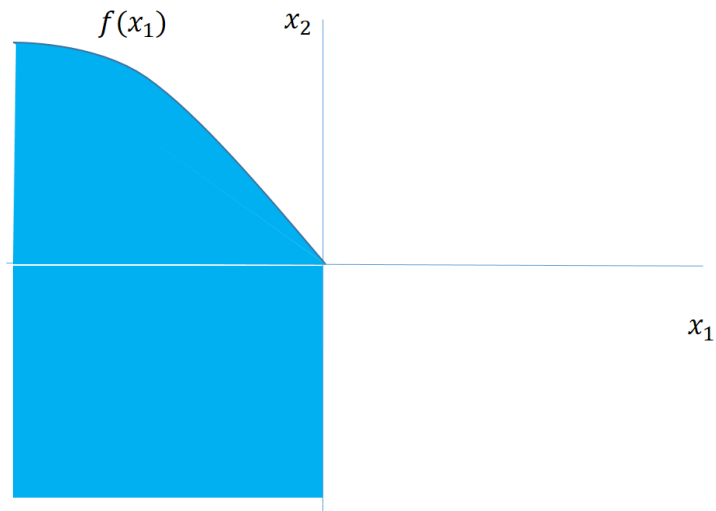
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- Would mean **using** 3 units of good 1 to produce 2 units of good 2 and 1 unit of good 3
- A firm is simply defined by its production set $Y \subset \mathbb{R}^L$
- This is the set of **feasible** production vectors for the firm

- A simple example with $L = 2$
- The firm uses commodity 1 to produce commodity 2
 - i.e. x_1 is always (weakly) negative
 - Call good one 'labor' and good 2 'sausages'
- The maximal amount of good 2 that can be produced given consumption of good 1 is given by $f(x_1)$
- But the firm can always 'throw away' good 2
- Then the production set looks like this....

The Production Set



- Often we will assume that for a firm commodities are split into **inputs** and **outputs**
 - Outputs are produced in positive amounts (q_1, \dots, q_M)
 - Inputs are used in positive amounts (z_1, \dots, z_{L-M})
- One special case is the one in which there is only 1 output
 - This is essentially the only case that we will deal with
- In this case we can define the production set using the **production function**

$$f(z_1, \dots, z_N)$$

- This is the maximal amount of q that can be produced with inputs z_1, \dots, z_N
 - When dealing with production functions it will be more convenient to treat the z 's as positive numbers
 - so

$$f(z_1, \dots, z_N) = \max \{x \mid (x, -z_1, \dots, -z_N) \in Y\}$$

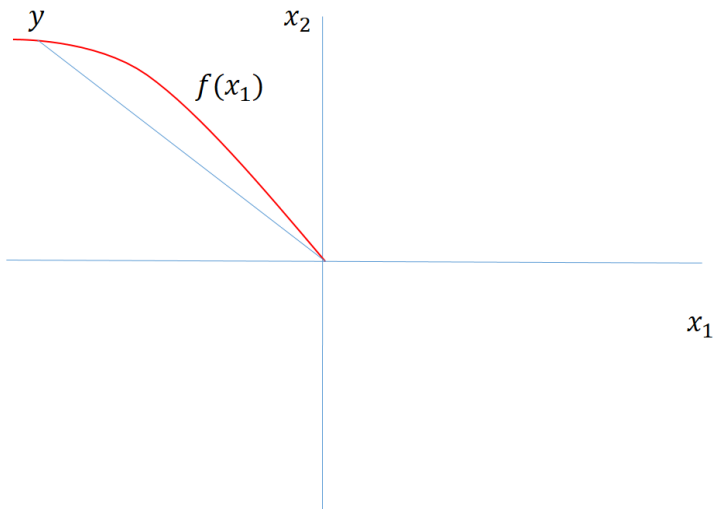
Properties of the Production Set

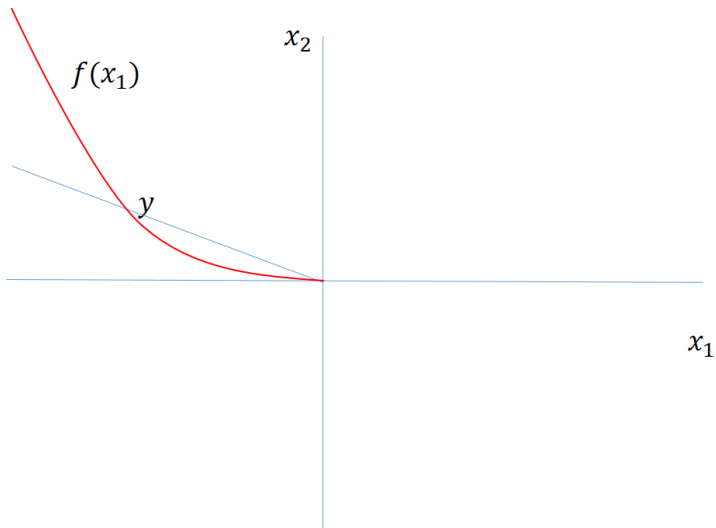
- Here are some properties that we might want the production set to have

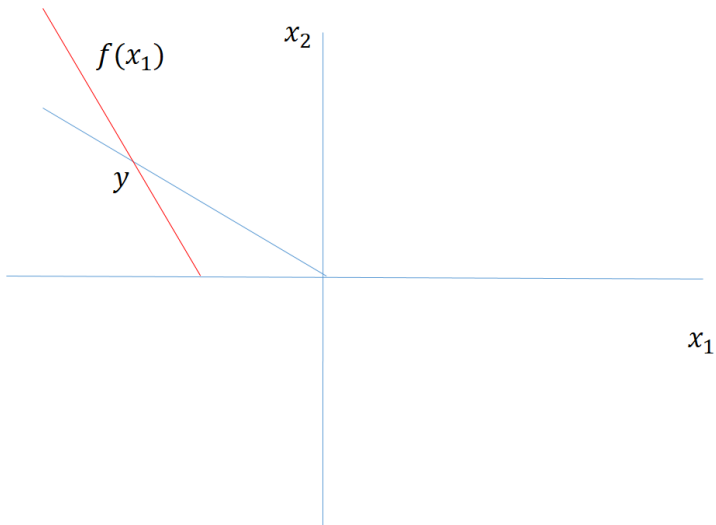
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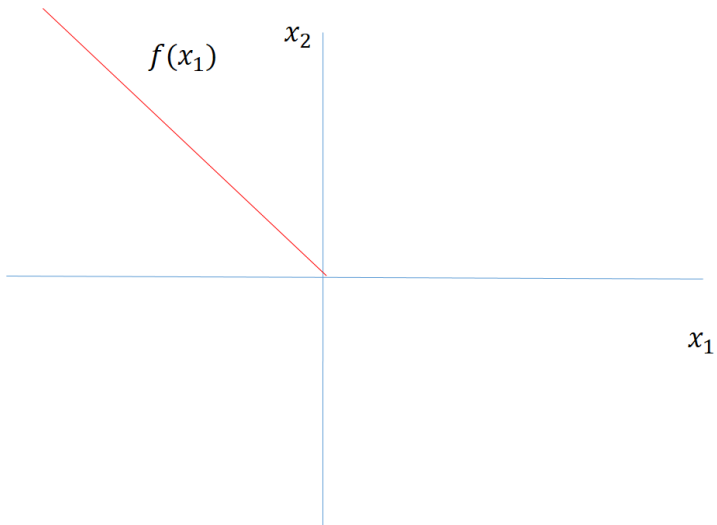
- Here are some properties that we might want the production set to have
- ① **Y is nonempty and closed.** This will help ensure that there is (a) something to study and (b) optimization problems involving the firm will have a solution
- ② $Y \cap \mathbb{R}_+^L = 0$. This says (a) that the firm can do nothing and (b) that there is no free lunch
- ③ **Free disposal - i.e. if $y \in Y$ and $y' \leq y$ then $y' \in Y$.** Firms can always dispose of any commodity at zero cost.

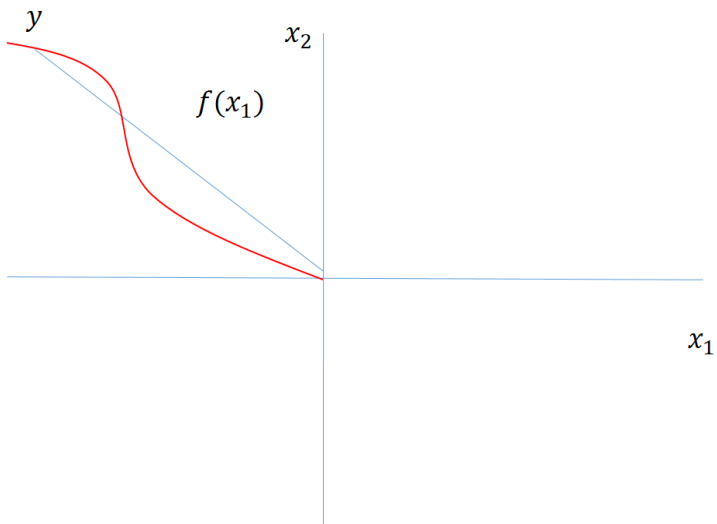
- A crucial characteristic of the production function is its **returns to scale**
 - Broadly speaking, does a firm become more or less efficient as it produces more output
- ① **Non-increasing returns to scale:** if $y \in Y$ then $\alpha y \in Y$ for $\alpha \in [0, 1]$
- ② **Non-decreasing returns to scale:** if $y \in Y$ then $\alpha y \in Y$ for $\alpha \geq 1$
- ③ **Constant returns to scale:** if $y \in Y$ then $\alpha y \in Y$ for $\alpha \geq 0$











- One assumption that is very handy (as usual) is convexity
- What type of returns to scale are necessary for Y to be convex?

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- Is this sufficient for Y to be convex?
- Not if there are multiple inputs
- Also requires that mixtures of inputs are more efficient than extremes
 - Analogous with consumption

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- So far we have used Y to define a firm in the same way that we used preferences to define a consumer
- But we have not defined an optimization problem for the firm!
- In the end, we will want firms to maximize profits
- But before that it is going to be extremely useful to define the **cost minimization problem**

- So far we have used Y to define a firm in the same way that we used preferences to define a consumer
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Definition

Consider a firm which produces 1 output using L inputs. Let $w \in \mathbb{R}_{++}^L$ be the vector of input prices. The cost minimization problem is

$$\begin{aligned} & \min_{z \geq 0} z \cdot w \\ & \text{subject to } f(z) \geq q \end{aligned}$$

Let $c(q, w)$ be the cost function and $z(q, w)$ be the factor demand correspondence

- Note
- ① We are assuming that the firm is a **price taker** - i.e. it treats the prices of inputs as fixed
- ② You have seen this problem before!
 - It is the expenditure minimization problem!
 - In fact, often we use the same functional forms for technology and preferences, such as Cobb Douglas
 - However, in this case the problem makes a lot more sense, and the cost function is easier to interpret than the expenditure function

- Let's start off with the case of 1 input
- In this case the cost minimization problem is easy!
- For any q , we have

$$\begin{aligned}z(q, w) &= f^{-1}(q) \\c(q, w) &= wf^{-1}(q)\end{aligned}$$

- That was boring!
 - Note that either we are assuming that f is strictly monotonic, or z may be a correspondence

- However, we can still learn something about the relationship between the cost function and the production function
- Define marginal costs as the derivative of the cost function with respect to q
- As

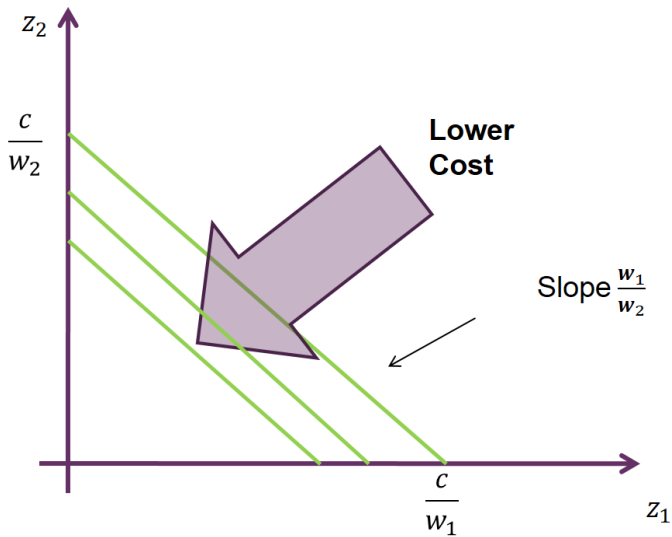
$$\begin{aligned}c(q, w) &= wf^{-1}(q) \\ \Rightarrow \frac{\partial c(q, w)}{\partial q} &= \frac{w}{\frac{\partial f(z)}{\partial z} \Big|_{z=z(q, w)}}\end{aligned}$$

- So, if $\frac{\partial f(z)}{\partial z}$ is **increasing** then $\frac{\partial c(q, w)}{\partial q}$ is **decreasing**
- So if f is concave (i.e. returns to scale are decreasing) then c is convex (i.e. marginal costs are increasing)
- if f is convex then c is concave
- If f is linear then so is c

The Case of Multiple Inputs

- The case of multiple inputs is more interesting!
- Now there are multiple different collection of inputs that will generate the same output
- Have to choose the cheapest one
- Let's start with some pictures in 2 dimensions
- First, we need iso-cost lines
 - This is what we are trying to minimize

Iso-Cost Lines

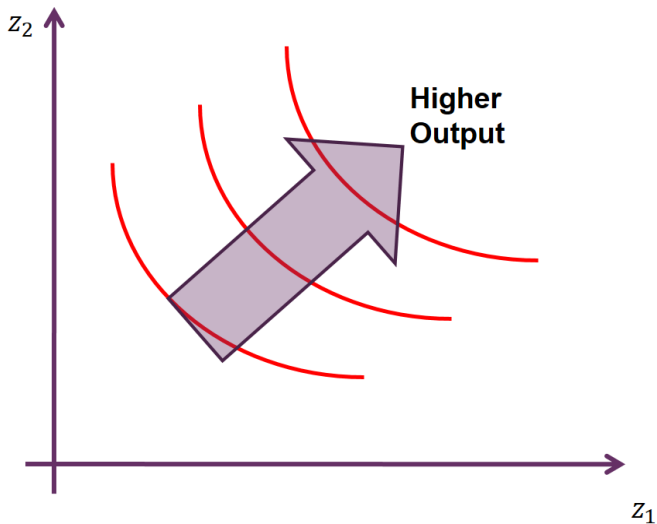


- Next we need the constraint
- i.e. the set

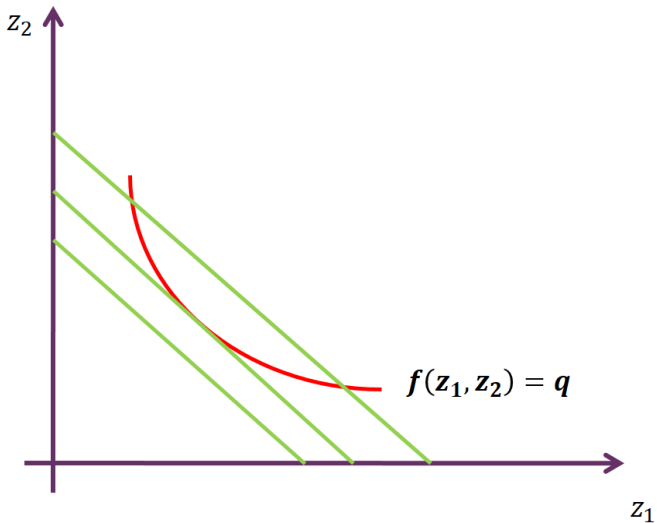
$$\{z | f(z) \geq q\}$$

- Clearly the shape of this is going to depend on the production function
 - Assume that f is weakly monotonic
 - If Y is convex then the iso output lines will be convex

Iso-Output Lines



- So we know what a solution is going to look like.....



- Assuming an interior solution it means that the slope of the iso-cost line is the same as that of the iso-output line
- Slope of the iso cost line is

$$\frac{w_1}{w_2}$$

- Iso-output line defined by

$$\begin{aligned} f(z_1, z_2) &= q \\ \Rightarrow \frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial z_2} dz_2 &= 0 \\ \Rightarrow \frac{dz_2}{dz_1} &= -\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} \end{aligned}$$

- This is the **marginal rate of technical substitution between** z_1 and z_2
 - The rate at which the firm can trade off z_1 and z_2 keeping output constant

- As with the consumer's problem we can set up the Lagrangian

$$\sum_L z_l w_l - \lambda(f(z) - q) - \sum_L \mu_l (-z_l)$$

- And get the solution

$$w_l \leq \lambda \frac{\partial f}{\partial z_l} \text{ and } w_l = \lambda \frac{\partial f}{\partial z_l} \text{ if } z_l > 0$$

- Here the Lagrangian really approach really comes into its own
- What is λ ?

- Here the Lagrangian really approach really comes into its own
- What is λ ?
- Recall the envelope theorem

$$\frac{\partial c(w, q)}{\partial q} = \frac{\partial f(z(w, \bar{q}) : w, q)}{\partial q} - \lambda \frac{\partial g_n(z(w, q) : w, q)}{\partial q}$$

- Recall that $\frac{\partial g_n(z(w, q) : w, q)}{\partial q}$ is the change in the value of the constraint with respect to q
 - In this case it is equal to 1
- $\frac{\partial f(z(w, q) : w, q)}{\partial q}$ is the direct impact of a change in q on the objective function
 - In this case equal to zero

- So

$$\frac{\partial c(w, \bar{q})}{\partial q} = \lambda$$

- It is the change in the object function with respect to a change in the constraint
- i.e. it is how $c(q, w)$ changes with q
- i.e. it is the marginal cost!
- Rather than solving for the function c and then differentiating with respect to q we can take the marginal cost straight from the Lagrangian!

Properties of Demands and Costs

- Now we will list some properties of z and c
- I'm not going to provide proofs
 - Easy adaptation of the proofs from consumption case
- Assume throughout that Y is closed and satisfies the free disposal property

- **Property 1:** z is a homogenous of degree 0 in w

Properties of Demands and Costs

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Properties of Demands and Costs

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- **Property 2:** c is homogenous of degree 1 in w and nondecreasing in q
- **Property 3:** c is a concave function of w
- **Property 4:** If the set $\{z | f(z) \geq q\}$ is convex the $z(w, q)$ is a convex set. If it is a strictly convex set then $z(w, q)$ is unique

- **Property 5:** (Shephard's lemma) If $z(w, q)$ is unique then $c(w, q)$ is differentiable with respect to w and

$$\frac{\partial c(w, q)}{\partial w_l} = z_l(w, q)$$

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- **Property 6:** If f is homogeneous of degree 1 (i.e. constant returns to scale) then c and z are homogenous of degree 1 in q

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- **Property 6:** If f is homogeneous of degree 1 (i.e. constant returns to scale) then c and z are homogenous of degree 1 in q
- **Property 7:** If f is concave then c is convex in q

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- Now we are in a position to define the ultimate goal of the firm: **prrrrrrrrrprofit!**
 - Not the only possible assumption, but a very standard one
- We will assume that the firm is a price taker on the output side as well
 - Output can be sold at a constant price p
- This is a 'perfect competition' assumption
- You will come across other alternatives later in the course
 - Monopoly
 - Oligopoly

Definition

Consider a firm which produces 1 output using L inputs. Let $w \in \mathbb{R}_{++}^L$ be the vector of input prices and p' be the output price. Let $p = (p', w)$ be the vector of all prices. The profit maximization problem is

$$\max_{z \geq 0} p'f(z) - w \cdot z$$

with $\pi(p)$ being the associated profit function and $y(p)$ the set of vectors in Y that maximize profit

- What do the first order conditions look like?

$$\frac{\partial f}{\partial z_l} \leq \frac{w_l}{p'}, \text{ with } \frac{\partial f}{\partial z_l} = \frac{w_l}{p'} \text{ if } z_l > 0$$

- Marginal product of an input z_l is equal to its price (in terms of output)
- Note that if Y is convex, these first order conditions are also sufficient

Properties of Profit Functions and Supply Correspondences

- **Property 1:** y is a homogenous of degree 0

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Properties of Profit Functions and Supply Correspondences

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- **Property 2:** π is homogenous of degree 1
- **Property 3:** π is a convex function
- **Property 4:** If Y is convex then y is convex. If Y is strictly convex then y is unique

Properties of Profit Functions and Supply Correspondences

- **Property 1:** y is a homogenous of degree 0
- **Property 2:** π is homogenous of degree 1
- **Property 3:** π is a convex function
- **Property 4:** If Y is convex then y is convex. If Y is strictly convex then y is unique
- **Property 5:** (Hotelling's Lemma) If $y(p)$ is unique, then π is differentiable at p and

$$\frac{\partial \pi(p)}{\partial p'} = y(p)$$

Properties of Profit Functions and Supply Correspondences

- What about comparative statics?
- What happens to the supply of outputs and demand for inputs as their own prices change

Properties of Profit Functions and Supply Correspondences

- What about comparative statics?
- What happens to the supply of outputs and demand for inputs as their own prices change
- It turns out that they are well behaved
 - Output is (weakly) increasing in output prices
 - Input demand is (weakly) decreasing in its own price
- This is because we are basically solving for 'compensated' demand functions, so the law of compensated demand holds

- It should be fairly easy to see that profit maximization problem is the same as

$$\max_{q \geq 0} p'q - c(w, q)$$

- And choosing $z \in z(w, q)$
- What are the first order conditions here?

- It should be fairly easy to see that profit maximization problem is the same as

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$$p' = c'(q)$$

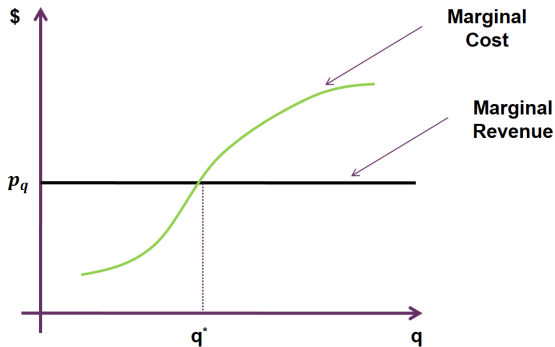
marginal revenue = marginal cost

- What about second order conditions?
- Requires

$$c''(q) \geq 0$$

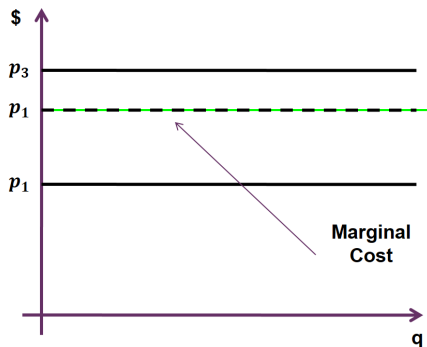
- i.e. marginal costs to be increasing
 - i.e. decreasing returns to scale
- This defines three basic cases

Decreasing Returns to Scale



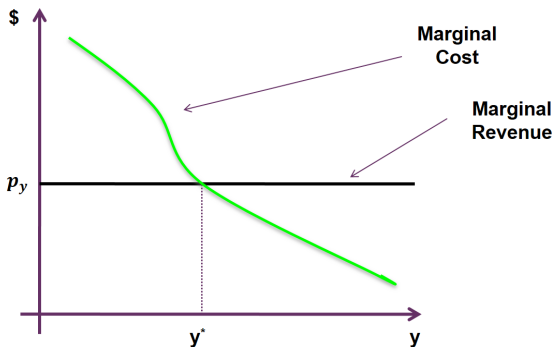
- (Typically) Interior solution
- Supply upward sloping
- Average cost below marginal cost

Constant Returns to Scale



- (Typically) corner solution
- Supply a step function (zero below marginal cost, infinite at marginal cost)
- Average cost equal to marginal cost

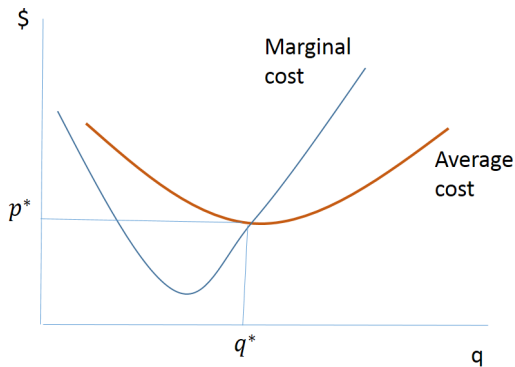
Increasing Returns to Scale



- Corner solution
- Supply a step function (zero if price below the lowest possible marginal cost, infinite otherwise)
- Average cost higher than marginal cost

Increasing Returns to Scale

- A popular textbook example is a case when returns to scale are initially increasing then decreasing



- Note that
 - The marginal cost curve crosses the average cost curve at its nadir
 - For any $p < p^*$ at any level of output, average cost less than price

$$\Rightarrow p - AC < 0$$

$$\Rightarrow pq - c(q) = \pi < 0$$

- so firm will produce 0
- For $p > p^*$ there are potentially two points at which $c'(q) = p$
- The one on the left is a local minimum
- The one on the right is the global maximum
- So the supply curve is given by the marginal cost curve above p^*