The Quantum Hall Effect

References:

H. Stormer, The Fractional Quantum Hall Effect, Nobel Lecture, December 8, 1998
R.B. Laughlin, Physical Review B 23, 5632 (1981)
Charles Kittel, Introduction to Solid State Physics
R.B. Laughlin, Fractional Quantization, December 8th, 1998

The integral quantum Hall effect was discovered in 1980 by Klaus von Klitzing, Michael Pepper, and Gerhardt Dordda. Truly remarkably, at low temperatures (~ 4 K), the Hall resistance of a two dimensional electron system is found to have plateaus at the exact values of h/ie^2. In the above expression, i is an integer, h is Planck's constant and e is the electron charge. At the same time, in the applied magnetic field range where the Hall resistance shows the plateaus, the magnetoresistance (i.e., the resistance measured along the direction of the current flow) drops to negligible values.

Two years later, the even more intriguing fractional quantum Hall effect was discovered by Horst L. Störmer, Daniel C. Tsui, and Arthur C. Gossard. When cooled down below ~2 K, the Hall resistance of the 2D electron systems shows plateaus at the values of $h/\nu e^2$, where ν is a fraction such as $1/3$, $1/7$, $2/3$, $4/5$ and so on. The value $R_H \approx 25.812$ kΩ for $i = 1$, the quantum of resistance, became the new world's resistance standard in 1990.

We are thankful to Professor Horst Stromer, who kindly agreed to direct the setting up of the first integral quantum Hall effect experiment designed especially for the education of undergraduate students. We also wish to thank Alexander Elias, the remarkable undergraduate student who was in charge of this project. In the future, we hope we can expand the range of applications of this experiment to observe also the fractional quantum Hall effect.
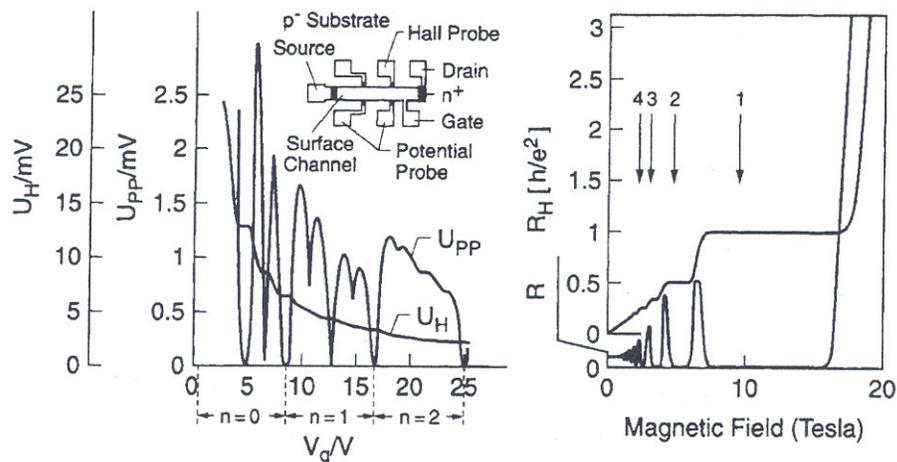
Figure 1: Left: original data of the discovery of the integral quantum Hall effect. Right: new data

On the left hand side of Fig. 1 is shown the original data of the discovery of the integral quantum Hall effect. The data was taken using a Metal Oxide Semiconductor Field Effect Transistor (MOSFET). In the experimental setup shown in the inset, by adjusting the gate voltage one can change the career density in the sample. The plateaus of the Hall voltage and the minima of the magneto resistance are distinguishable as the gate voltage is varied. Newer data taken using a GaAs/AlGaAs heterojunction confirms the quantization of the Hall resistance, as shown on the right hand side of Fig. 2.

The standard Hall effect

Figure 2
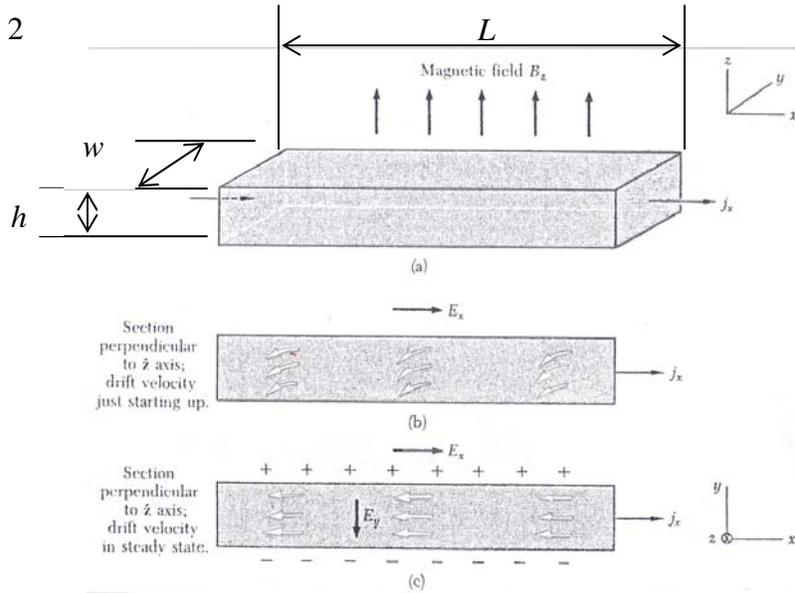
Figure 2 shows the standard geometry for observing the Hall effect, with axes labeling used throughout the experiment. A specimen of rectangular cross – section is placed in a magnetic field, B_z. An electric field E_x is applied between the end electrodes (the source and the drain) and makes an electric current of current density j_x flow through the sample. The magnetic Lorentz force acting on the moving electric charges causes the electrons to accumulate on one face of the specimen, leaving an excess of positive charges on the other face. These charges produce the transverse electric field E_y (Hall field). E_y increases as the electric charge builds up on the sides of the sample, until the electrostatic force with which it acts on the electrons balances the Lorentz force created by the magnetic field. Disregarding the signs, $eE_y = evB_z$, or $E_y = vB_z$, with v the drift velocity of electrons along the x direction. The Hall resistance is defined as $R_H = V_H/I_x$. V_H is the Hall voltage, $V_H = wE_y$. The magnitude of the current flowing in the x direction is $I_x = Aj_x$. $A = wh$ is the area of the cross-section of the sample perpendicular to the direction of the current. $j_x = n_{3D}ev$, where n_{3D} is the volume

density of free electrons in the sample. Therefore, $R_H = \dfrac{wvB}{whn_{3D}ev} = \dfrac{B}{ne}$. $n = n_{3D}h$ is the density of electrons per unit area ($n_{3D} = N/whL$ and $n = N/wL$, with N the number of electrons in the sample).

Reading: Motion in magnetic field, Chapter 6 in Kittel

Devices used for observing the quantum Hall effect.

The quantum Hall effect is a phenomenon occurring in a 2-dimensional electron gas. Originally, it was observed using a metal-oxide-semiconductor transistor (MOSFET). In a MOSFET, the electrons are trapped at the interface between silicon, which is a semiconductor, and silicon oxide, an insulator. The electric field E_z, applied perpendicular to the interface through the metal gate, pushes the electrons strongly against the silicon oxide side of the interface, as shown in Fig. 3a). Therefore, the motion along the z direction is strongly constrained. On the other hand, by varying the gate voltage, one can control the density of electrons at the interface, which makes the MOSFET a field effect transistor.

We study the quantum Hall effect using GaAs - AlGaAs hetero-junctions. The two-dimensional electron system is formed at the interface between the GaAs and AlGaAs and has a fixed density per unit area, n. The quantum Hall effect is observed by changing the magnetic field, rather than changing the density of the sample. The electrons prefer the GaAs side of the interface to the AlGaAs side because it has a slightly lower Fermi energy (about 300 meV difference). A typical GaAs - AlGaAs sandwich is shown in Fig. 3b). Both GaAs and AlGaAs are semiconductors, with almost identical lattice constants. Therefore, the interface is of high quality, without defects and stresses.

Silicon impurities are introduced in the AlGaAs layer during the deposition procedure. The AlGaAs layer is about 0.5 μm thick, while the impurities sit at about 0.1 μm from the interface with GaAs. The silicon impurity has one more electron on the outer shell than the gallium atom. This electron will move in the crystal and eventually fall in the GaAs. The positive silicon ions pull the mobile electrons against the AlGaAs at the GaAs – AlGaAs interface. The combined effect of the difference in Fermi energies of the two semiconductors and of the electrostatic pull of the silicon ions is to create a triangular-shaped quantum-well at the interface. Along the z direction, the electrons are trapped in the discrete states of this quantum-well. At low temperatures, only the lowest energy level of the quantum-well is occupied, as shown in Fig. 3c). On the other hand, the electrons can freely move in the xy plane.

Reading: H. Stormer, The Fractional Quantum Hall Effect, Nobel Lecture, December 8, 1998: Introduction, Two-Dimensional Electron Systems, and Modulation Doping

Figure3

a)

b)

c)

2DES
(Wavefunctions)
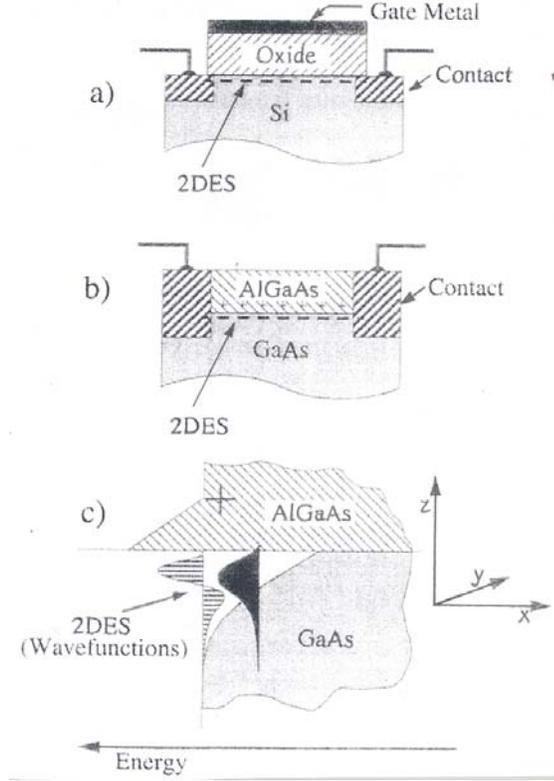
General discussion of the Landau levels in a crystal

We discuss first the 3D motion of the electrons in an applied magnetic field and then apply the results for the 2D electron system.

The vector potential of a uniform magnetic field $B_{\hat{z}}$ is $\vec{A} = -By\hat{x}$ in the Landau gauge. The Hamiltonian of the free electron without spin is:

$$H = -(\hbar^2/2m)(\partial^2/\partial y^2 + \partial^2/\partial z^2) + (1/2m)[-i\hbar\,\partial/\partial x - eyB]^2 ,$$

where e is the electronic charge, m is the mass of the electron, and $\hbar = h/2\pi$ is Planck's constant.

We look for eigenfunctions of this Hamiltonian satisfying the wave equation $H\Psi = E\Psi$. The current flows through the sample in the $\hat{x}$ direction, while the electrons are free to move along the direction of the magnetic field, the $\hat{z}$ direction. Therefore, the wave function is chosen of the form:

$$\psi = \chi(y)\exp[i(k_x x + k_z z)],$$

where k_x and k_z are the wave-vectors characterizing the motion of electrons along the $\hat{x}$ and $\hat{z}$ directions.

By applying the wave equation to this wave function, one obtains the following equation for $\chi(y)$:

$$H = (\hbar^2/2m)\,d^2\chi/dy^2 + [E - (\hbar^2 k_z^2/2m) - \frac{1}{2}m\omega_c^2(y - y_0)^2]\chi = 0,$$

where $\omega_c = eB/m$ and $y_0 = -\hbar k_x/eB$.

The above identity can be interpreted as the wave equation of a 1D harmonic oscillator with frequency ω_c and centered at position y_0. The energies of this oscillator are quantized:

$$E_n = \left(n + \frac{1}{2} \right) \hbar \omega_c + \hbar^2 k_z^2 / 2m, \text{ with } n = 0, 1, 2....$$

The average position of this oscillator (or of the electron) is y_0, which depends on the applied magnetic field. Therefore, the overall effect of the magnetic field is to shift the electronic states in the $\hat{y}$ direction, perpendicular to the direction of the current. This electron displacement gives rise to the Hall voltage.

In a 2D electron system, when the motion of electrons along the field direction z is negligible, the allowed electronic states are equally spaced, with energies given by: $E_i = \left(i - \frac{1}{2} \right) \hbar eB / m$. The counting starts from 1 for convenience, $i = 1,2,3,....$ These are the so-called Landau levels. In a real crystal, one has to replace m, the bare mass of electrons, with m^*, their effective mass. $\omega_c = eB / m^*$ is the cyclotron frequency of the electrons in the crystal.

The situation is more complex, actually. There are two types of states available to the 2D electrons, which are graphically represented in Fig. 4: localized states and extended states. The localized states are the bound states formed around the defects in the crystal. The extended states are the states carrying the electric current across the sample. The cyclotron motion of the electrons is superimposed on the motion of electrons along the direction of the current, characterized by the drift velocity v. The extended states can bend around localized states or pass through 3D impurities, without the current carried by these states being affected.

Figure 4: Motion of the electrons in a 2D plane, in the presence of 3D impurities or defects in the crystal. The Hall voltage is measured across the direction of the current.
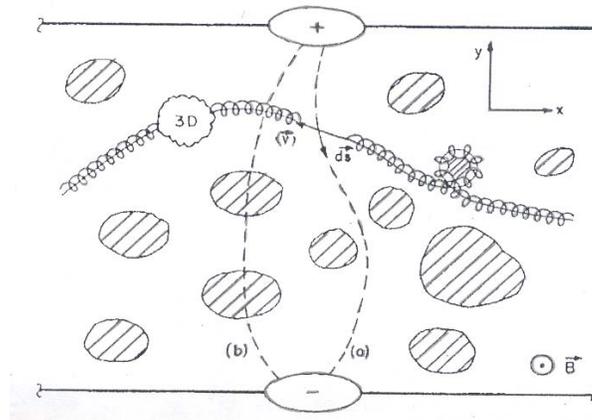
Qualitative discussion in terms of the Fermi energy of the system

The density of states of an ideal 2D electron system, with no impurities or other imperfections creating localized states, in a magnetic field B applied perpendicular to the plane of electrons, consists of a series of equidistant δ

functions when plotted as a function of energy. This is shown in Fig. 5a). If we neglect the electron spin, the degeneracy of each magnetic level is $d = eB/h$. d measures the number of electronic orbits that can be packed in each Landau level per cm^2 of sample.

Figure 5: Density of states as a function of energy:
  a) of an ideal 2D crystal;
  b) of a 3D crystal or a 2D crystal at lower fields;
  c) of a 2D crystal under higher applied fields.
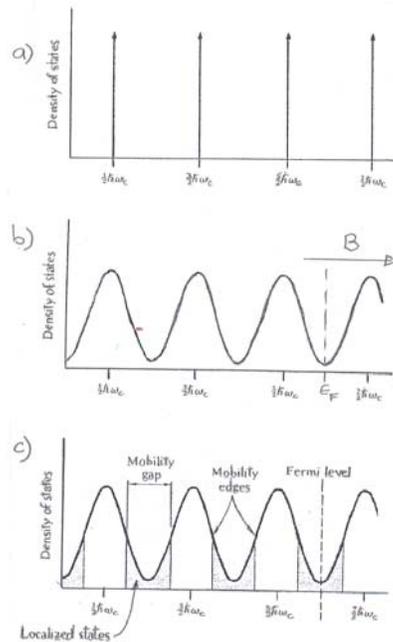
The Fermi energy of an ideal 2D electron system at T = 0 K is determined only by the number of electrons in the sample, when there is no magnetic field applied. In a magnetic field, in order to satisfy the minimum energy principle, the electrons will fill the Landau levels starting from the lowest energy level. In most of the cases, a number of Landau levels will be completely filled, and the highest Landau level will be partially occupied. The Fermi energy is the energy of the last Landau level in which electrons reside. At very low temperatures (4 K) and high magnetic fields (up to 10 T), the thermal excitations between Landau levels are negligible. If the magnetic field is increased, the capacity of each Landau level will increase, and electrons from higher levels will drop to lower Landau levels until they are filled again. If one continues to increase the magnetic field, the highest Landau level will be depleted, while all the levels below are exactly filled. The Fermi energy of the system will drop suddenly. We expect this sudden change of the Fermi energy to be obvious in the properties of the sample, such as conductivity, heat capacity, etc.

Reading: Chapter 9 in Kittel, Experimental Methods in Fermi Surface Studies.

We can write the condition for an exact number of Landau levels to be filled as $n = id$, where n is the electron density per cm^2 of sample, $d = eB/h$ is the degeneracy of the Landau levels, and i is an integer, $i = 1,2,3,\dots$. For a sample of given density, n, there is a discrete set of magnetic fields, B_i, $i = 1,2,3,\dots$, for which this condition is satisfied, $B_i = (nh/e)/i$. The Hall resistance, generally

defined as $R_H = B/ne$, takes a discrete set of values at the magnetic fields B_i, $R_H = h/ie^2$. These values correspond precisely to the plateaus of the Hall resistance determined experimentally.

The drop of the magnetoresistance of the sample around the magnetic fields B_i can be also explained by the complete occupancy of the Landau levels. The longitudinal resistance is determined by the amount of energy an electron loses by inelastic scattering. In order for scattering to occur, an electron must be able to jump to an empty allowed state. The extended states inside of the occupied Landau levels are filled by other electrons, while the states in the unoccupied Landau levels are inaccessible thermally, the energy associated with the thermal motion being much smaller than the separation between the Landau levels. Therefore, the scattering between electrons is not permitted, and the resistivity drops to negligible values.

In a 3D crystal the Landau levels have the extended appearance as a function of the energy shown in Fig. 5b). This is due to the motion of electrons along the direction of the applied magnetic field. Under low applied magnetic fields, when the separation between Landau levels is small, the picture is valid also for the real 2D systems.

As in the ideal case above, when the magnetic field is increased, the spacing between the Landau levels becomes larger, but also the capacity of each Landau level increases. We can picture the density of states at the Fermi level oscillating as the Landau levels advance in energy with increasing B field. The resistivity of the sample is determined by the derivative of the density of electronic states at the Fermi level. The resistivity has therefore an oscillatory behavior. This oscillating feature of the magnetoresistance is called the Shubnikov de Hass effect and has been observed in all 3D metals and also in the 2D metals at lower magnetic fields.

Under large magnetic fields, the energy dependence of the density of states becomes similar to Fig. 5c). The Landau levels are narrower in energy, resembling the ideal case, while between the Landau levels only localized electronic states exist, forming the so-called mobility gap. We can picture this situation as the case in which the cyclotron velocity becomes considerably larger than the drift velocity of the electrons. The difference from the ideal case is that the electrons occupy also the localized states between the Landau levels. When an integer number of Landau levels are completely filled, the Fermi level lies inside the mobility gap. But because the states at the Fermi level are localized, they do not contribute to conductivity. Therefore, the resistivity of the sample along the direction of the current drops to zero.

According to the previous theoretical discussion, the quantum Hall effect is produced only at the very special values of the magnetic field, $B_i = (nh/e)/i$. The larger plateau regions in the Hall resistance and the corresponding minima of magnetoresistance are readily explained by the presence of the localized states. The localized electrons act as reservoirs of carriers: for example, as the magnetic field is increased, the electrons from the localized states will complete the filling of the Landau levels, maintaining them at full capacity for larger field intervals.

The magnetoresistance correspondingly drops to negligible values in these regions of magnetic field.

Reading:

H. Stormer, The Fractional Quantum Hall Effect, Nobel Lecture, December 8, 1998: The Hall Effect and The Integral Quantum Hall Effect.
Chapter 19 in Kittel, Magnetoresistance in a Two-Dimensional Channel
R.B. Laughlin, Physical Review B 23, 5632 (1981)

Effects of the electron spin
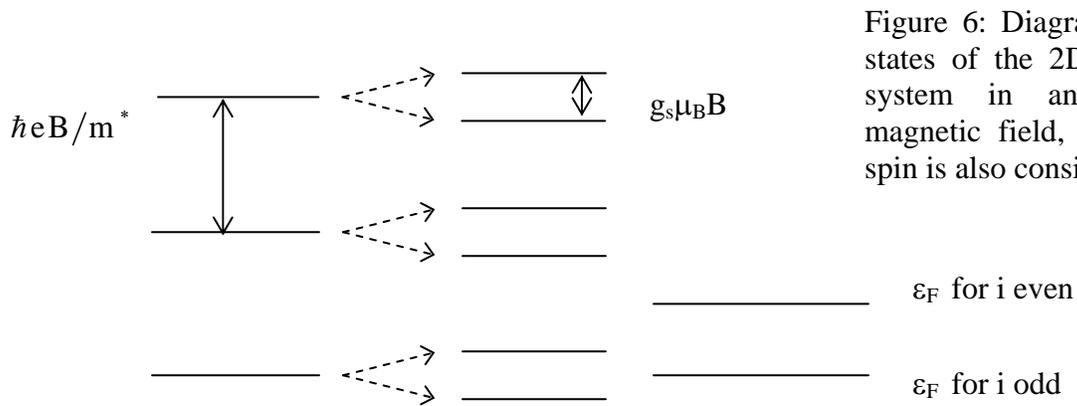
Figure 6: Diagram of the states of the 2D electron system in an applied magnetic field, when the spin is also considered

$\hbar eB/m^*$

$g_s\mu_B B$

ε_F for i even

ε_F for i odd

The analysis of the quantum Hall effect becomes more complex when one takes the spin into consideration. Under the influence of the magnetic field, the Landau level split, as shown in Fig. 6. The Zeeman splitting of the levels is given by $g_s\mu_B B$. g_s is the electron gyromagnetic factor, which is much smaller in the solid as compared to the gyromagnetic factor g_s of the free electron. μ_B is the Bohr magneton, $\hbar e/2m$. Because the small effective mass of the electrons in the solid strongly enhances the cyclotron frequency, the energy gap between the levels formed by Zeeman splitting is approximately a factor of 70 smaller than the energy gap between two successive Landau levels in the absence of the spin. The degeneracy of each of the Landau levels stays the same.

The quantum Hall effect is obtained at the same values of the magnetic field B_i, which indicate that an integer number of Landau levels are exactly filled. When an odd number of levels are filled, the Fermi level of the system lies in the small gap between the Landau levels produced by Zeeman splitting, as shown in Fig. 6 for i = 1. It is easy to thermally activate the electrons close to the Fermi energy to the next (i = 2, in this case) Landau level. Therefore, the magnetoresistance of the sample will show a non-zero component in the case of oddly numbered B_i fields. Correspondingly, in the plot of the Hall resistance versus magnetic field, the odd numbered plateaus of the Hall resistance will not clearly appear. On the other hand, when an even number of levels are filled, the

Fermi energy lies in the large mobility gap between successive levels. The electrons do not have enough thermal energy to jump to the next Landau level, and we recover the quantization of the Hall resistance and the drop of the magnetoresistance.

Note on the Fractional Quantum Hall Effect

At lower temperatures (less than 2 K), features of the fractional quantum Hall effect can be distinguished. The fractional quantum Hall effect is characterized by the plateaus of the Hall resistance at values $R_H = h/\nu e^2$, where ν takes fractional values. The magnetoresistance drops correspondingly to negligible values.

Reading: H. Stormer, The Fractional Quantum Hall Effect, Nobel Lecture, December 8, 1998: The Fractional Quantum Hall Effect

The integral quantum Hall effect can be explained solely by the filling of the Landau levels. Each Landau level has a certain capacity to accept electrons, which depends on the magnetic field B. By changing the magnetic field, we change the ability of each Landau level to accommodate electrons. When there is a match between the capacity of the Landau levels and the number of electrons in the sample, an integer number of Landau levels are exactly filled, and the integral quantum Hall effect is produced.

In the case of the fractional quantum Hall effect (FQHE), one must account also for the correlations between the electrons. The electronic charge will be distributed as to create the state with minimal total energy, which is the most favorable state. But the situation is more complex, there being a significant difference between the odd numbered fractions, such as 1/3, and the even numbered fractions, such as 1/2, or 1/4.

Let's consider only the 1/2 and 1/3 cases, which are the simplest. The FQHE is produced in these cases at very large fields. The effect of a strong field perpendicular to a 2D mobile charge system is to create vortices (regions in the 2D system where the density of charge is 0). This is just a way to minimize the free energy of the system. There is a fixed amount of magnetic field flux passing through each vortex (the magnetic flux quantum, $\Phi_0 = h/e$). If one increases the magnetic field, the number of vortices will increase.

The 1/2 and 1/3 FQHE occur when there are more vortices than electrons: twice as many vortices in the case of the 1/2 FQHE, and three times as many vortices in the case of the 1/3 FQHE. The essence of the FQHE is that the electrons can "pair" with the vortices to find the state with minimal energy. One can understand the expression "an electron sits on a vortex" thinking at the quantum mechanical average of the electron position. Similar to the case of the

Hydrogen atom: the ground state is spherically symmetric around the hydrogen nucleus, so that the average electron position is the nucleus. But we know that the electron is actually moving around the nucleus.

In the case of the 1/2 FQHE, one electron pairs with two vortices to form the most energetically favorable state. In the case of the 1/3 FQHE, one electron pairs with three vortices. There are symmetries associated with this pairings. An electron is a fermion, and the vortices also have fermion like properties - the wave function associated with a vortex is antisymmetric when we interchange the position of two vortices. Think of the vortices in the water: they have a rotation direction, they are not characterized only by position. The quantum mechanics of vortices in the electron sea is also more complicated, because the electrons around the vortices created by the magnetic field are not static.

A composite particle formed by an electron (a fermion) paired with an even number of vortices is a fermion (the total wave function is multiplied by -1 when we interchange two composite particles of this type: we multiply it by -1 to account for the electron and also by an even number of -1 to account for the vortices). A composite particle formed by an electron paired with an odd number of vortices is a boson (the total wave function is unchanged when we interchange two composite particles of this type: we multiply the wave function with -1 to account for the electron and odd number of -1 to account for the vortices).

Bosons and fermions obey different statistics: the bosons condense at the lowest temperature on the lowest energy level available. On the other hand, two fermions cannot occupy the same quantum mechanical level. So they fill successive energy levels, starting from the lowest energy level, as to minimize the free energy of the system.

Therefore, at odd numbered fractions such as 1/3, the composite particles are bosons, and they form the Bose condensate, characterized by an energy gap. The scattering between the composite particles is prohibited, which produces the prominent drop in the magnetoresistance. At even numbered fractions such as 1/2, 1/4, the composite particles are fermions and they do not condense. The filling of states is similar to the filling of electronic states up to the Fermi level in the absence of the magnetic field, and the ground state does not show an energy gap. The FQHE does not appear in the Hall resistance measurements and the drop of the magnetoresistance is barely distinguishable. But the composite particles (made of an electron and an even number of vortices) exist, and their formation is generated by the tendency of the system to occupy the state with minimum energy available.

If the magnetic field is slightly shifted from the magnetic field producing the perfect 1/2 or 1/3 state, single vortices, unpaired with electrons, will appear in the electron sea. A single vortex moving in the electron sea produces distortions of the electron charge. Amazingly, a single vortex appears to be carrying with it

1/2, 1/3, etc, of the electronic charge. The new composite particles with 1/2e, 1/3e charge are vortices moving through the electron sea and carrying with them an average charge of 1/2e, 1/3e. These are the fractionally charged quasiparticles of the FQHE. A very simplistic comparison is an analogy with the phonons in a crystal: an electron moving in a crystal produces lattice distortions, which propagate along with the electron. The composite particle is the phonon, which has an enhanced mass compared to the electron.

The quasiparticles of the FQHE carry exactly 1/2 or 1/3 etc of the charge because the FQHE is a many particle effect, which occurs at very special ratios between the number of vortices created by the magnetic field and the number of electrons in the system. The correlations between the components of the many body system formed by the 2D electrons in a high magnetic field plays an important role in determining what the properties of the fractionally charged quasiparticles are.