The Stationary Distribution of a Stochastic Clearing Process

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This research grew out of an investigation of utilization in capacity expansion. The utilization at any time is the demand divided by the capacity. When there is uncertainty about the evolution of demand, it is appropriate to model the demand as a stochastic process, and thus the utilization also becomes a stochastic process. It was found that a utilization stochastic process associated with exponentially growing stochastic demand is closely related to the stochastic clearing processes introduced and investigated by Stidham. Interest in the impact of uncertainty on utilization led to this study of the impact of uncertainty on the stationary distribution of a stochastic clearing process. Stidham showed for a large class of clearing processes that the stationary distribution is never the uniform distribution, which is characteristic of deterministic models with continuous linear input. Here it is shown for a larger class of clearing processes that the stationary distribution is always stochastically less than or equal to the uniform distribution in the sense of second-order stochastic dominance (characterized by the expected value of all nondecreasing concave functions). For various special cases, stronger stochastic order relations are established. For a related capacity expansion model, it is shown that greater uncertainty lowers the expected utilization.

RECENTLY Stidham (1974, 1977) and Serfozo and Stidham (1978) have investigated a class of stochastic processes called clearing processes, which have many applications in the study of stochastic input-output systems, e.g., queues, dams and inventories; see Stidham (1977). A clearing process represents the content of a service system that is intermittently and instantaneously cleared. A clearing process starts at zero and each time it exceeds a level q it instantaneously returns to zero and starts over. A typical sample path is shown in Figure 1. The purpose of this paper is to present some comparison results for clearing processes. These comparisons help describe the impact of uncertainty in stochastic input-output systems.

This research was motivated by a project to study the factors affecting utilization in capacity expansion. We define the utilization at time t as the proportion of the capacity that is being used at time t, i.e., the demand divided by the capacity: $U(t) = D(t)/C(t)$, $t \geq 0$. Suppose that demand grows expotentially but randomly, i.e., let $D(t) = e^{Y(t)}$, $t \geq 0$,

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where $Y(t)$ is some stochastic process. Suppose that successive capacity expansions occur at the epochs when demand exceeds capacity. Let the new capacity at each expansion be a constant multiple γ of the demand at the expansion epoch. Since $\log U(t) = Y(t) - \log C(t)$, it turns out that the process $V(t) \equiv \log U(t) + \log \gamma$ is a clearing content process with respect to the input process $Y(t)$ and the clearing level $\log \gamma$. To see this, let C_n be the epoch of the nth capacity expansion and note that $V(t) = Y(t) - Y(C_n)$, $C_n \leq t < C_{n+1}$, and

$$\begin{aligned}\log U(t) &= Y(t) - \log C(t) = Y(t) - \log \gamma D(C_n),\\ &= Y(t) - Y(C_n) - \log \gamma = V(t) - \log \gamma,\ C_n \leq t < C_{n+1}.\end{aligned}$$

(See Section 1 for more detail.) Hence, the utilization process $U(t)$ has a stationary distribution if and only if the clearing content process $V(t)$

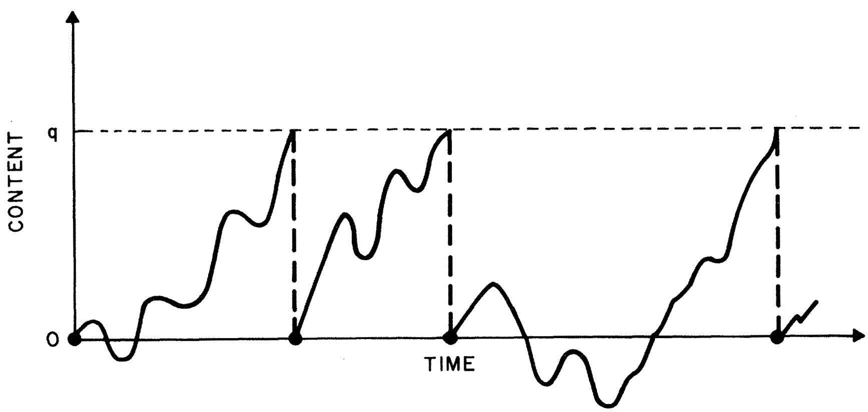

Figure 1. A typical sample path of a stochastic clearing process.

has a stationary distribution. Moreover, if $V^*(0)$ has the stationary distribution of the content process and $U^*(0)$ has the stationary distribution of the utilization process, then $U^*(0)$ has the distribution of $\gamma^{-1}e^{V^*(0)}$. Consequently, many (but not all) of the stochastic comparisons here for clearing processes carry over to these utilization stochastic processes. Where the comparisons do carry over, we see that uncertainty tends to lower the expected utilization. For the model in which demand is geometric Brownian motion, i.e., where the demand grows exponentially and the associated stochastic process Y is Brownian motion as in Section 4 here, we can quantify the decrease in expected utilization caused by uncertainty. This utilization application is discussed in Section 5.

Another application is the classical economic lot size inventory model. Our analysis shows how the deterministic model is affected by uncertainty about demand. In the deterministic model let the cumulative demand for a single divisible commodity in the interval $[0, t]$ be μt, $\mu > 0$. Let the designated policy be to order an amount q, which is delivered instantaneously, whenever the inventory level $X(t)$ reaches 0. The deterministic inventory level process is not itself the content process of a clearing system, but the closely related process $\{q - X(t), t \geq 0\}$ is. Clearly, neither $X(t)$ nor $q - X(t)$ has a limiting distribution as $t \to \infty$ because these processes are periodic. However, it is easy to see that both processes have a unique stationary probability distribution, namely, the uniform distribution over the interval $[0, q]$. The uniform distribution is the unique randomization of the inventory level at time 0 such that the distribution of the inventory level is unchanged for all $t \geq 0$. At the same time, it assigns to each subset of $[0, q]$ a probability equal to the long-run proportion of time the process spends in that subset.

For the economic lot size model, it is natural to ask what is the impact of uncertainty about demand. In particular, suppose that the deterministic economic lot size model is used to generate a policy even though there is in fact some uncertainty about demand. The stochastic clearing processes represent reasonable alternatives to the continuous linear deterministic demand. It is thus interesting to ask how the stationary distribution of the inventory level in the stochastic case compares with the uniform stationary distribution in the deterministic case. The comparisons here imply that the inventory level will tend to be higher with uncertainty than without. Moreover, in some cases, such as with the Brownian motion input processes discussed in Section 4, the inventory level distribution increases with increasing uncertainty. Since the inventory level is obviously directly related to costs, we see one way uncertainty affects costs.

In fact, a major focus of the research in Serfozo and Stidham (1978), Stidham (1974, 1977) has already been on the relation between the stationary distribution and the uniform distribution which is characteristic of deterministic models with continuous linear input. A principal conclusion of Stidham (1974) was that for a large class of clearing processes the stationary distribution is never uniform (except in certain degenerate deterministic cases). The purpose of this paper is to establish some additional comparison results for the stationary distributions of clearing processes. We consider the clearing processes studied in Sections 3 and 4 of Stidham (1974), i.e., clearing processes with either a compound input process or an input process with stationary independent increments. However, we generalize by not requiring that the input process have nondecreasing paths. That is important because the sample paths of input processes are often not nondecreasing.

We show (Theorem 1 and Section 3) that the stationary content distribution is always stochastically less than or equal to the uniform distribution in the sense of second-order stochastic dominance, which is characterized by the expected value of all nondecreasing concave real-valued functions; see Brumelle and Vickson (1975) and references there. It is easy to show by example that in general this ordering cannot be strengthened to first-order stochastic dominance, which is characterized by the expected value of all nondecreasing real-valued functions. However, if the batch sizes associated with a compound input process are nonnegative and the batch size distribution has a decreasing failure rate, then the stationary distribution is stochastically less than or equal to the uniform distribution in the first-order sense (Theorem 2). To obtain this, we apply a recent result of Brown (1980) which establishes conditions under which the renewal function is concave. The same first-order stochastic dominance relationship is also shown to hold for processes with stationary independent increments that have no positive jumps, e.g., Brownian motion (Theorem 4). We note (Theorem 3) that bounds can be obtained for the stationary content distribution associated with compound input processes by applying bounds for the renewal function in Barlow and Proschan (1975), Brown, and Lorden (1970). These bounds show that the stationary distribution is approximately uniform if the clearling level is sufficiently high.

In Section 4 we briefly discuss clearing systems with compound input processes in which the successive batch sizes form a Markov chain. We also obtain additional stochastic order relations for various special cases, including clearing systems with Brownian motion input processes. We conclude the paper in Section 5 with a brief discussion of the application to utilization in capacity expansion. We provide additional supporting details in three appendices which have been omitted to save space; they are available from the author.

1. REGENERATIVE CLEARING PROCESSES

As in Section 2 of Stidham (1974), we begin by considering clearing content processes that are regenerative with respect to the clearing epochs. Let the input process $\{Y(t),\ t \geq 0\}$ have $Y(0) = 0$ and sample paths which are right-continuous with left limits but not necessarily nondecreasing. Let the system be immediately cleared whenever the content exceeds the level q, which brings the content back to zero. Let $\{C_n,\ n \geq 1\}$ be the sequence of clearing epochs, defined by $C_n = \inf\{t \geq C_{n-1}\colon Y(t) > Y(C_{n-1}) + q\}$, $n \geq 1$, with $C_0 = 0$. As part of the regenerative assumption, we assume $\{C_n,\ n \geq 1\}$ is i.i.d. and $EC_1 < \infty$. Then the content process $\{V(t),\ t \geq 0\}$ is defined by $V(t) = Y(t) - Y(C_n)$, $C_n \leq t < C_{n+1}$, $t \geq 0$. Let $\{V^*(t),\ t \geq 0\}$ be the stationary version of the content process, which exists as in Stidham (1974) because of the regenerative structure. The stationary version is a strictly stationary stochastic process that is obtained by placing the origin appropriately in the interior of a clearing cycle. The formal construction is described on p. 94 of Brown and Ross (1972) and p. 1276 of Miller (1972). Because the input process sample paths are right-continuous with left limits, existence is guaranteed. In fact, we shall only be concerned with the one-dimensional marginal distribution, i.e., the real-valued random variable $V^*(0)$. The distribution of $V^*(0)$ can also be described as

$$\begin{aligned} P(V^*(0) \le y) &= \lim_{t\to\infty} t^{-1} \int_0^t 1_{(-\infty,y]}(V^*(s))ds \\ &= \lim_{t\to\infty} t^{-1} \int_0^t 1_{(-\infty,y]}(V(s))ds \\ &= E \int_0^{C_1} 1_{(-\infty,y]}(V(s))ds/EC_1 \\ &= ET(y, q)/ET(q, q), \qquad y \le q, \end{aligned} \tag{1}$$

where $1_A(x)$ is the indicator function of set A and $T(y, q)$ is the total time Y spends in the set $(-\infty, y]$ before first hitting the set (q, ∞). The first two terms on the right side of (1) are the asymptotic distributions of the stationary version and the original content process, respectively; compare p. 171 of Serfozo and Stidham. Equation (1) follows from the regenerative structure; Proposition 5.9 of Ross (1970). If the distribution of C_1 is nonlattice, then $V(t)$ converges in distribution as $t \to \infty$ to $V^*(0)$; see Theorem 3.1 of Miller. However, we do not assume the distribution of C_1 is nonlattice. If $Y(t)$ is a Markov process, then $ET(y, q)$ is just the potential measure of the set $(-\infty, y]$ associated with a unit left charge at 0 for the process $Y(t)$ modified to be absorbing when it hits the set (q, ∞); for further discussion, see Section 4. (These potential theory concepts are not used in this paper.)

Recall that a random variable X_1 is stochastically less than or equal to another random variable X_2 in the sense of first-order stochastic dominance, denoted by $X_1 \le^{st} X_2$, if $P(X_1 > t) \le P(X_2 > t)$ for all t. It is well known that $X_1 \le^{st} X_2$ if and only if $Ef(X_1) \le Ef(X_2)$ for all nondecreasing real-valued functions f for which the expectations exist. Let V_d^* be a random variable uniformly distributed on the interval $[0, q]$. (The notation is intended to suggest the stationary distribution in the deterministic case.) As an immediate consequence of (1), we see that $V^*(0) \le^{st} V_d^*$ if and only if $ET(y, q)/y \ge ET(q, q)/q$, $y < q$. To get this stochastic order, it obviously suffices to have $ET(y, q)/y$ be nonincreasing in y, as noted in a less general setting on p. 94 of Stidham (1974).

A random variable X_1 is stochastically less than or equal to another random variable X_2 in the sense of second-order stochastic dominance, denoted by $X_1 \leq_2 X_2$, if $Ef(X_1) \leq Ef(X_2)$ for all nondecreasing concave real-valued functions. It is known that $V^*(0) \leq_2 V_d^*$ if and only if

$$\int_0^t P(V^*(0) \geq y)dy \leq \int_0^t P(V_d^* \geq y)dy, \qquad 0 \leq t \leq q; \qquad (2)$$

see page 104 of Brumelle and Vickson. From (1), we see that $V^*(0) \leq_2 V_d^*$ if and only if

$$t^2 ET(q, q)/(2q) \leq \int_0^t ET(y, q)dy, \qquad 0 \leq t \leq q. \qquad (3)$$

In the following sections we shall exhibit conditions under which these inequalities hold.

2. COMPOUND INPUT PROCESSES

In this section we consider clearing systems with compound input processes as in Section 3 of Stidham (1974). Briefly, this means that at time $\tau_1 + \cdots + \tau_n$ input batches of size σ_n arrive. The basic assumption in Stidham (1974) is that $\{\tau_n\}$ and $\{\sigma_n\}$ are independent sequences of i.i.d. nonnegative random variables, but we only assume that τ_n is nonnegative for all n at the outset. With $M(t) = \max\{n \geq 0\colon \tau_1 + \cdots + \tau_n \leq t\}$, $t \geq 0$, the input process $\{Y(t), t \geq 0\}$ is defined as $Y(t) = \sum_{n=1}^{M(t)} \sigma_n$, $t \geq 0$. Let $S_n = \sigma_1 + \cdots + \sigma_n$, $n \geq 1$, and $M_n = \max\{S_1, \cdots, S_n\}$. Let $N(x, y)$ be the number of indices for which the partial sums are less than or equal to x before the partial sums first exceed y, i.e.,

$$N(x, y) = \#\{n \geq 1\colon S_n \leq x, M_n \leq y\}, \qquad x \leq y. \qquad (4)$$

Let $Z(y, q) = N(y, q) + 1$, $y \leq q$, and $U(y) = Z(y, y)$, $y \geq 0$. Obviously $U(y)$ is the first passage time or inverse process associated with the sequence $\{S_n\}$. The key random variable $T(y, q)$ can be represented here as

$$T(y, q) = \sum_{n=1}^{U(q)} \tau_n 1_{\{j:S_{j-1} \leq y\}}(n), \qquad y \leq q, \qquad (5)$$

where $S_0 = 0$ and $1_A(n)$ is the indicator function of the set A. If the sequence $\{\tau_n\}$ is i.i.d. and independent of the sequence $\{\sigma_n\}$, then $T(y, q)$ clearly has the same distribution as $\sum_{n=1}^{Z(y,q)} \tau_n$ and $ET(y, q) = EZ(y, q)E\tau_1$, which implies that the distribution of the stationary clearing content $V^*(0)$ is independent of the distribution of τ_1. An example in which $\{S_n\}$ is a Markov chain is discussed in Section 4.

Another simplification occurs if $\sigma_i \geq 0$ for all i. Then $T(y, q) = T(y, y) = \sum_{n=1}^{U(y)} \tau_n$, $y \leq q$. If $\{\tau_n\}$ is i.i.d. with $E\tau_1 < \infty$ and $U(y)$ is a stopping time relative to $\{\tau_n\}$, then $ET(y, y) = EU(y)E\tau_1$ by Wald's equation, p. 137 of Chung (1974) and the distribution of $V^*(0)$ again is independent of the distribution of τ_1. An important example occurs when $\{(\tau_n, \sigma_n), n \geq 1\}$ is a sequence of i.i.d. random vectors, but τ_n and σ_n may be dependent. (This observation and example are due to Richard Serfozo.)

For our main result, we use the following elementary lemma, see p. 521 of Lorden.

LEMMA 1. *If $\{\sigma_n\}$ is i.i.d., then $U(x + y) \leq^{st} U_1(x) + U_2(y)$ where $U_1(x)$ and $U_2(y)$ are independent random variables with the distribution of $U(x)$ and $U(y)$ respectively.*

Proof. If $J(x) = S_{U(x)} - x$, then $U(x + y)$ is distributed as $U_1(x) + U_2(y - J(x))$, where $U(z) = 0$ for $z \leq 0$ and U_1 and U_2 are independent. Since $U(z)$ is nondecreasing in z and $J(x) \geq 0$, $U_2(y - J(x)) \leq U_2(y)$.

COROLLARY. *If $\{\sigma_n\}$ is i.i.d., then $EU(x + y) \leq EU(x) + EU(y)$ for all $x, y \geq 0$.*

For our main result, we also need some additional consequences of the subadditivity obtained in the corollary above.

LEMMA 2. *If $h(x)$ is a real-valued function such that $h(x + y) \leq h(x) + h(y)$ for all $x, y \geq 0$, then*

$$\text{(a)}\quad xh(x) \leq 2 \int_0^x h(s)ds,\ x \geq 0.$$

and

$$\text{(b)}\quad h(x) \leq 2x \int_0^1 h(s)ds,\ x \geq 1.$$

Proof. (a) By the subadditivity assumption,

$$h(x) = x^{-1} \int_0^x h(x)ds \leq x^{-1} \int_0^x [h(s) + h(x - s)]ds = 2x^{-1} \int_0^x h(s)ds.$$

(b) Let $z = x - [x]$, where $[x]$ is the greatest integer less than or equal to x. By the subadditivity assumption and part (a),

$$\begin{aligned} h(x) &\leq ([x] - 1)h(1) + h(1 + z) \\ &\leq 2([x] - 1) \int_0^1 h(s)ds + \int_0^1 [h(s) + h(1 + z - s)]ds \\ &\leq 2([x] - 1) \int_0^1 h(s)ds + \int_0^z h(s)ds + 2 \int_z^1 h(s)ds + \int_0^z h(1 + s)ds, \end{aligned}$$

and

$$\int_0^z h(1+s)ds \le \int_0^z [h(1)+h(s)]ds$$
$$\le zh(1) + \int_0^z h(s)ds$$
$$\le 2z\int_0^1 h(s)ds + \int_0^z h(s)ds.$$

The proof is completed by combining these inequalities.

Conditions for $EU(y)$ to be finite when $\{\sigma_n\}$ is i.i.d. follow from the general theory of random walks; see Section 8.4 of Chung. It suffices for σ_1 to have a finite positive mean, but it is necessary and sufficient to have

$$\textstyle\sum_{n=1}^{\infty} n^{-1}P(S_n \le 0) < \infty. \tag{6}$$

Condition (6) does not allow $E\sigma_1 \le 0$; it covers the case in which a mean does not exist.

Henceforth, we make the following assumptions. We assume that $\{(\sigma_n, \tau_n), n \ge 1\}$ is an i.i.d. sequence of random vectors with $E\tau_1 < \infty$ and σ_1 satisfying (6). We also suppose that either $\sigma_1 \ge 0$ or $\{\sigma_n\}$ is independent of $\{\tau_n\}$. These assumptions guarantee that the content process $V(t)$ is regenerative with respect to the clearing epochs and $ET(y, q) < \infty$ for $y \le q$, so that the stationary version of the content process exists and is given by (1). Moreover, the distribution of $V^*(0)$ in (1) is independent of the distribution of τ_1, because

$$ET(y, q) = EZ(y, q)E\tau_1, \ y \le q. \tag{7}$$

Our main result is that the stationary content $V^*(0)$ is always stochastically less than or equal to V_d^* (the random variable that is uniformly distributed on $[0, q]$) in the sense of second-order stochastic dominance.

THEOREM 1. $V^*(0) \le_2 V_d^*$.

Proof. By (3) and (7), it suffices to show that $t^2EZ(q, q) \le 2q\int_0^t EZ(y, q)dy$, $0 \le t \le q$. Let $h(y) = EU(y)$. Since $h(y) \le EZ(y, q)$, $0 \le y \le q$, and $h(q) = EZ(q, q)$, it suffices to show that $h(q) \le (2q/t^2)\int_0^t h(y)dy$, $0 \le t \le q$. After a change of variables ($s = y/t$), it suffices to show that $h(q) \le 2(q/t)\int_0^1 h(s/t)ds$, $0 \le t \le q$. Let $g(s) = h(s/t)$ and $x = q/t$. Now it suffices to show that $g(x) \le 2x\int_0^1 g(s)ds$, $x \ge 1$. However, notice that g inherits the subadditivity property from h obtained via the Corollary to Lemma 1. Hence, the proof is completed by applying Lemma 2(b).

COROLLARY. $EV^*(0) \leq EV_d^* = q/2$.

Example 1. This example shows that the expected value ordering in Theorem 1 need not hold when the input process is generalized. We consider a clearing system with a compound input process having $\{\sigma_n\}$ and $\{\tau_n\}$ independent, $\{\tau_n\}$ i.i.d., but $\{\sigma_n\}$ not i.i.d. For a given q, let $\sigma_{3k-2} = q - 1$ and $\sigma_{3k-1} = \sigma_{3k} = 1$ for all $k \geq 1$ with probability one. It is easy to see that the content process V is regenerative, but $EV^*(0) = (2q - 1)/3 > q/2$ for $q > 2$.

Example 2. This example shows that stronger stochastic order relations between $V^*(0)$ and V_d^* need not hold. In particular, the order relation

$$Ef[V^*(0)] \leq q^{-1} \int_0^q f(y)dy$$

need not hold for all nondecreasing convex real-valued functions f. To substantiate this claim, let $q = 1$ and consider a compound input process with constant batch size $\sigma_1 = 1$. Then $P(V^*(0) = 0) = P(V^*(0) = 1) = ½$. Now consider $f(x) = e^x$. Then $Ef[V^*(0)] = (1 + e)/2 \geq e - 1 = Ef[V_d^*]$.

Example 3. This example shows that a stochastic order relation between two batch size distributions does not imply any ordering between the associated stationary distributions. Let σ_{i1} be the size of the first batch in the ith clearing system, $i = 1, 2$. Suppose $P(\sigma_{11} = q + 1) = 1$ and $P(\sigma_{21} = q + 1) = P(\sigma_{21} = -1) = ½$. Then $\sigma_{11} \geq^{st} \sigma_{21}$ and $P(V_1^*(0) = 0) = 1$. It is easy to see that $V_1^*(0)$ and $V_2^*(0)$ are not comparable with stochastic order. For sufficiently large q, $EV_1^*(0) \leq EV_2^*(0)$ too.

We now show that it is possible to make stronger comparisons if we make additional assumptions about the batch size distribution. Recall that a nonnegative random variable X is said to have a DFR (decreasing failure rate) distribution if $P(X > t) > 0$ for all t and $P(X \geq s + t | X \geq t)$ is nondecreasing in t for all s; see Barlow and Proschan.

THEOREM 2. *If the batch size σ_1 is nonnegative with a DFR distribution, then $V^*(0) \leq^{st} V_d^*$.*

Proof. By Theorem 3 of Brown, the renewal function associated with the sequence of batch sizes, $EU(y) = ET(y, q)/E\tau_1$, is concave under the DFR assumption, which implies that $EU(y)/y$ is decreasing in y ($-U$ is star-shaped, p. 106 of Barlow and Proschan). As noted in Section 1, this implies the stochastic order.

Remark. The stochastic ordering for the case of constant failure rate, i.e., the exponential distribution, was noted by Stidham (1974) in Example 3.5. It is also easy to see that the stationary distribution associated with exponential batch size is stochastically increasing as the mean of the exponential distribution decreases.

We now bound the stationary distribution using the first two moments of the batch size distribution.

THEOREM 3. *If* $\sigma_1 \geq 0$, $E\sigma_1 = \mu_1$ *and* $E\sigma_1^2 = \mu_2$, *then*

$$y/(q + \mu_2\mu_1^{-1}) \leq P(V^*(0) \leq y) \leq y/q + \mu_2/q\mu_1, \qquad 0 \leq y \leq q.$$

Proof. By Wald's identity, we have the well known ordering

$$EU(y) \geq y\mu_1^{-1}, \qquad y \geq 0,$$

and, by Theorem 1 of Lorden,

$$EU(y) \leq y\mu_1^{-1} + \mu_2\mu_1^{-2}, \qquad y \geq 0.$$

Simply apply these inequalities in the equilibrium distribution as given in (1) and (7) with $EU(y) = EZ(y, y) = EZ(y, q)$.

Remarks. (1) Refinements of these bounds are available under additional assumptions; see p. 171 of Barlow and Proschan, Brown and references there.

(2) The lower bound in Theorem 3 is valid if the batch size σ_1 is not constrained to be nonnegative because

$$EZ(y, q)/EZ(q, q) \geq EZ(y, y)/EZ(q, q).$$

In this case μ_2 can be replaced by $E\{(\sigma_1^+)^2\}$, see Lorden. Obviously, more interesting bounds when σ_1 is not constrained to be nonnegative would follow from bounds on $EZ(y, q)$.

(3) As noted by Stidham (1974) the elementary renewal theorem implies that $V^*(0)$ approaches the uniform distribution as $q \to \infty$ when $\sigma_1 \geq 0$. Theorem 3 provides a bound on the rate of convergence.

3. INPUT PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

Many properties of clearing systems with compound input processes easily extend to clearing systems with stochastically continuous input processes having stationary independent increments either because such input processes are the continuous analogs of compound input processes or because such input processes can be represented as the limit of a sequence of compound input processes; see Section 4 of Stidham (1974) and Section IX.6 of Gikhman and Skorohod (1969). The stochastic continuity entails no important loss of generality and provides some regularity properties; see pp. 304–306 of Breiman (1968).

The stationary distribution of the clearing process with an input process having stationary independent increments is again given by (1).

If the input process has nondecreasing paths, then $T(y, q)$ coincides with $T(y, y)$ whose expected value reduces to the sojourn measure $W(y)$ in Stidham (1974). In order to have $T(y, q) < \infty$ for $y \le q$, we assume that

$$\textstyle\sum_{k=1}^{\infty} k^{-1}P(Y(k) \le 0) < \infty. \tag{8}$$

As with condition (6), (8) implies that the expected number of steps before the random walk $\{Y(k), k \ge 1\}$ exceeds q is finite, which in turn obviously implies that $ET(q, q)$ is finite. A sufficient condition for (8) is of course $E|Y(1)| < \infty$ and $EY(1) > 0$. It is easy to see that Lemma 1 and, thus, Theorem 1 extends to this setting. Hence, $V^*(0) \le_2 V_d^*$.

It is elementary but significant that convergence of the input processes (even weak convergence in the function space setting) does not imply convergence of the equilibrium distributions. However, for any given input process Y with stationary independent increments, it is clearly possible to choose a special sequence of compound input processes $\{Y_n\}$ such that $V_n^*(0)$ converges in distribution to $V^*(0)$ as $n \to \infty$. In this way, we could obtain an alternate proof of the analog of Theorem 1. Given the input process Y with stationary independent increments, the sequence of compound input process $\{Y_n, n > 1\}$ is constructed by setting $\tau_{nk} = 2^{-n}$ and $\sigma_{nk} = Y(\mathrm{K}2^{-n}) - Y((k-1)2^{-n})$, $k \ge 1$ and $n \ge 0$.

We conclude this section with another stochastic order relationship.

THEOREM 4. *If the input process $Y(t)$ with stationary independent increments has no positive jumps (called the spectrally negative case), then $V^*(0) \le^{st} V_d^*$.*

Proof. First, for any $n \ge 1$, $T(y, y)$ is distributed as the sum of n i.i.d., random variables with the distribution of $T(y/n, y/n)$. Hence, $ET(cq, cq) = cET(q, q)$ first for all rationals c and then for $c = y/q$ by taking limits. Finally, $ET(y, q) \ge ET(y, y) = (y/q)ET(q, q)$.

An example of an input process covered by Theorem 4 is Brownian motion with a positive draft. However, we exhibit the stationary distribution of the content process in this case in the next section and obtain an even stronger ordering.

4. SPECIAL CASES

In this section we first return to compound input processes. We assume that $\{\tau_n\}$ is i.i.d. and independent of $\{\sigma_n\}$, but we do not assume $\{\sigma_n\}$ is i.i.d. We assume that an independent copy of $\{\sigma_n\}$ is used for each clearing cycle. Within each cycle, we assume $\{S_n\}$ is a Markov chain with stationary transition probabilities. As we have seen in Section 2, without loss of generality we can assume $P(\tau_1 = 1) = 1$, which makes the content process a Markov chain.

For simplicity of exposition, we assume the Markov chain $\{S_n\}$ is on the integers as in Kemeny et al. (1966). Let the clearing level q be an integer. To analyze the clearing process, we make the set of states $\{k{:}k > q\}$ absorbing. We assume that the Markov chain with these designated absorbing states is absorbing (see Kemeny et al., p. 112). By the strong Markov property (see p. 88) the content process V associated with this input process Y is regenerative and has a unique stationary distribution (also see Stidham [1974], Section 2). The stationary distribution is easily expressed in terms of the fundamental matrix N (Kemeny et al., p. 107), or the potential measure ν associated with a unit left charge at 0 (p. 192). Recall that

$$N = \textstyle\sum_{k=0}^{\infty} Q^k,$$

where Q is the transition matrix of the Markov chain restricted to the nonabsorbing states, $Q^0 = I$ and ν is the potential measure associated with the transition matrix Q and the left charge μ where $\mu(\{x\}) = \delta_{\{0\}}(x)$: $\nu(y, q) = \sum_{j=-\infty}^{y} N_{0j}$. In this Markov setting,

$$P(V^*(0) \leq y) = \nu(y, q)/\nu(q, q), y \leq q.$$

A special case of interest is a random walk on the integers. Then $Q_{ij} = p_{j-i}$ for some probability vector p and N_{ij} is the Green function (see Spitzer [1964], pp. 111, 274). It is easy to construct examples showing that the expected value ordering $EV^*(0) \leq q/2$ can fail in the Markov chain case, but it holds in the random walk case by virtue of Theorem 1.

There are some special cases for which stronger orderings can be established. First, for the simple random walk with $p_{-1} = p$, $p_1 = r$, $p + r = 1$ and $r > p$, it is not difficult to calculate the fundamental matrix N (see Appendices 1–3, available from the author):

$$N_{ij} = \begin{cases} (1/r)(1 + p/r + \cdots + (p/r)^{q-j}), & i \leq j \leq q, \\ (p/r)^{i-j} N_{ii}, & j < i \leq q. \end{cases} \tag{9}$$

It is then easy to see (Appendix 2) that the family of probability mass functions $\{P(V_r^*(0) = j), \tfrac{1}{2} < r \leq 1\}$ associated with a family of clearing systems indexed by the positive-step probability r satisfies the monotone likelihood ratio property (Ferguson [1967], p. 208), which implies that $V_r^*(0)$ is stochastically increasing in r; see Whitt (1980) for additional properties. In fact, $V_r^*(0)$ increases stochastically to $V_1^*(0)$, the stationary content level in the deterministic lattice case, having $P(V_1^*(0) = k) = (1 + q)^{-1}$, $0 \leq k \leq q$, and $EV_1^*(0) = q/2$; Example 3.6 of Stidham (1974).

We now observe that stochastic order also holds between $V_1^*(0)$ and the random variable $V^*(0)$ associated with any random walk that moves up at most one step at a time (and satisfies (6)). The following theorem is proved just like Theorem 4.

THEOREM 5. *If* $p_k = 0$ *for* $k \geq 2$, *then* $V^*(0) \leq^{st} V_1^*(0)$.

The results for the simple random walk in (9) obviously extend to Brownian motion because Brownian motion is the continuous analog of a random walk. (The connection is made precise in Appendix 3.) If $Y(t) = \mu t + \sigma B(t)$, where $B(t)$ is standard Brownian motion, $\mu > 0$ and $\sigma \geq 0$, then the expected first passage time to q is q/μ, independent of σ (see Karlin and Taylor [1975], p. 361), and $V^*(0)$ has the continuous density

$$f_{V^*(0)}(x) = \begin{cases} q^{-1}(1 - e^{-\lambda(q-x)}), & 0 \leq x \leq q, \\ q^{-1}(1 - e^{-\lambda q})e^{\lambda x}, & x \leq 0, \end{cases} \tag{10}$$

where $\lambda = 2\mu/\sigma^2$. This can be proved using (9) via a limit theorem for random walks (as shown in Appendix 3) or directly from diffusion theory; Theorem 2 of Puterman (1975) or Theorem 2 of Whitt (1973). The moments of $V^*(0)$ can be calculated from the moment generating function, which is

$$Ee^{sV^*(0)} = \lambda(e^{qs} - 1)/(qs(\lambda + s)), \qquad s \geq 0. \tag{11}$$

It is easy to see that the family of densities $\{f_{V_\lambda^*(0)}(x), \lambda > 0\}$ satisfies the monotone likelihood ratio property too. Hence, the random variable $V_\lambda^*(0)$ increases stochastically to the uniform distribution as $\lambda \to \infty$ ($\sigma \to 0$ and/or $\mu \to \infty$). The stochastic ordering $V_\lambda^*(0) \leq^{st} V_d^*$ also follows from Theorem 4.

With Brownian motion there are always excursions below zero in each clearing cycle, so one might think that is the reason for the ordering. However, the stochastic ordering with the uniform distribution is still valid if a reflecting barrier is put at the origin. It is not difficult to see that the stationary distribution with the reflecting barrier is just the conditional distribution $P(V_\lambda^*(0) \leq y \mid V_\lambda^*(0) \geq 0)$ which also increases stochastically to the uniform distribution as $\lambda \to \infty$.

5. UTILIZATION IN CAPACITY EXPANSION

As we mentioned in the introduction, this research was largely motivated by efforts to gain a better understanding of utilization in capacity expansion. In order to learn how to interpret utilization measurements, we have developed (jointly with H. Luss) models to describe how utilization is affected by various important factors. We now describe one such model that employs the clearing process with Brownian motion input that we have just considered at the end of Section 4.

Let the utilization at time t be the proportion of the capacity that is being used at time t, i.e., the demand divided by the capacity: $U(t) = D(t)/C(t)$, $t \geq 0$. Assume that a fixed proportion α (possibly zero) of the demand is needed as administrative spare to manage the system efficiently, for example, as a safety factor. Let successive capacity expansions occur at the epochs when demand plus administrative spare exceeds capacity. Let the new capacity at each expansion be a constant multiple γ of the demand plus administrative spare at the expansion epoch. Also assume that capacity is continuously retired at a rate ρ times the existing capacity; that is, when t is not an expansion epoch, $dC(t)/dt = -\rho C(t)$. Finally, assume that demand is geometric Brownian motion: $D(t) = D(0)e^{\mu t+\sigma B(t)}$, $t \geq 0$; pp. 357 and 363 of Karlin and Taylor. If time 0 is an expansion epoch, then the utilization at time t (before the next expansion) is given by

$$\begin{aligned} U(t) &= D(t)/C(t) = D(0)e^{\mu t+\sigma B(t)}/C(0)e^{-\rho t} \\ &= e^{(\mu+\rho)t+\sigma B(t)}/((1 + \alpha)\gamma), \qquad t \geq 0. \end{aligned} \tag{12}$$

The next expansion occurs at the random time τ defined by

$$\begin{aligned} \tau &= \inf\{t \geq 0\colon U(t) \geq (1 + \alpha)^{-1}\} \\ &= \inf\{t \geq 0\colon (\mu + \rho)t + \sigma B(t) > \log \gamma\}. \end{aligned} \tag{13}$$

From (12) and (13), we see that the retirement factor ρ can be combined with the demand drift term μ and the administrative spare factor α appears as the constant multiplicative term $(1 + \alpha)^{-1}$. Moreover, as outlined in the introduction, $\log U(t) + \log(\gamma(1 + \alpha))$ is a clearing content process with respect to the Brownian motion input process $Y(t) = (\mu + \rho)t + \sigma B(t)$ and the clearing level $\log \gamma$. In Section 4 we noted that the clearing content process associated with this input process Y has the unique stationary distribution (which is also the limiting distribution) given in (10), with $\lambda = 2(\mu + \rho)/\sigma^2$ now. Hence, the utilization process $\{U(t), t \geq 0\}$ has a unique stationary distribution which is stochastically increasing in λ. Since the utilization process has a limiting distribution, it is natural to interpret a utilization measurement (when this model is appropriate) as an observation from this limiting distribution.

Since $U^*(0) = [(1 + \alpha)\gamma]^{-1}e^{V^*(0)}$, we obtain the following simple product formula (which we have not been able to resist calling the Whitt-Luss utilization formula):

$$EU^*(0) = (\lambda/(\lambda + 1))(1/(1 + \alpha))((\gamma - 1)/(\gamma \log \gamma)), \tag{14}$$

where $\lambda = 2(\mu + \rho)/\sigma^2$. Formula (14) is obtained from (11) by setting $s = 1$ and $q = \log \gamma$. From (14), it is easy to see the impact of the factors μ, σ, ρ, α and γ on the expected utilization.

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